JENSEN'S NFU, 40 YEARS LATER

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Genealogy of NFU

- 1879: Frege's *Begriffsschrift*.
- 1901: Russell's $\{\mathbf{x} : \mathbf{x} \notin \mathbf{x}\}$.
- 1908: Russell's Ramified Type Theory RTT.
- 1925: Ramsey's Simple Types Theory TT.
- 1937: Quine's *New Foundations* NF.
- 1940/1951: Quine's *Mathematical Logic*.
- 1953: Rosser's *Logic for Mathematicians* .
- 1953: Specker's refutation of AC in NF.
- 1962: Specker's equiconsistency proof of NF withTT + Ambiguity.
- 1969: Jensen's consistency proof of NFU.

What's NF?

- The *language* of NF is $\{=, \in\}$.
- The *logic* of NF is classical first order logic.
- The *axioms* of NF are Extensionality and Stratified Comprehension.
- φ is *stratified* if there is an integer valued function f whose domain is the set of **all** variables occurring in φ, which satisfies the following two requirements:
- $f(v) + 1 = f(w)$, whenever $(v \in w)$ is a subformula of φ;
- $f(v) = f(w)$, whenever $(v = w)$ is a subformula of φ.

Basic facts about NF

- The formula $x = x$ is stratifiable, so there is a universal set **V** in **NF**.
- NF proves that **V** is a Boolean algebra!
- Cardinals and ordinals are defined in NF in the spirit of Russell and Whitehead.
- card(X):= $\{Y$: there is a bijection from X to $Y\}$.
- **Card** $:= \{\lambda : \exists X\ (\lambda = \text{card}(X))\}$ exists in NF.
- Similarly, the set of all ordinals **Ord** exists in NF.

Cantor's Theorem in NF

- What about Cantor's $\text{card}(\mathcal{P}(X)) > \text{card}(X)$ for $X = \mathbf{V}$?
- In the usual proof of Cantor's Theorem, given $f : X \to \mathcal{P}(X)$, we look at $\{x \in X : \neg(x \in f(x))\}$, whose defining equation is not stratifiable!
- $\text{USC}(X) = \{\{x\} : x \in X\}$ exists, and in the NF context, Cantor's theorem is reformulated as $\text{card}(\mathcal{P}(X)) > \text{card}(USC(X))$.
- For a cardinal λ, let $T(\lambda) := \text{card}(\text{USC}(X))$, where X is some (any) element of λ, and define (in the metatheory):
- $\kappa_0 := \text{card}(V)$;
- $\kappa_{n+1} := T(\kappa_n)$.
- $\text{NF} \vdash \kappa_0 > \kappa_1 > \cdots > \kappa_n > \cdots$

Cantorian and strongly Cantorian sets

- X is *Cantorian* if $\text{card}(X) = \text{card}(\text{USC}(X))$.
- X is *strongly Cantorian* if $\{\langle x, \{x\}\rangle : x \in X\}$ exists.
- Rosser's AxCount(Axiom of Counting): $\mathbb{N}$ is strongly Cantorian.
- $\mathbb{N} := \{\text{card}(X) : \text{fin}(X)\}$.
- $\text{fin}(X) :=$ "there is no injection from X into a proper subset of X".
- $\text{NF} \vdash \text{AxCount} \leftrightarrow$ "all finite sets are Cantorian".
- **Theorem (Orey, 1964)** $\text{NF} + \text{AxCount} \vdash \text{Con}(\text{NF})$.

Finite axiomatizability

- **Theorem (Hailperin, 1944).** NF *is finitely axiomatizable, and* $\mathrm{NF} = \mathrm{NF}_6$.
- **Theorem (Grishin, 1969).** $\mathrm{NF} = \mathrm{NF}_4$, *and* $\mathrm{Con}(\mathrm{NF}_3)$.
- **Theorem (Boffa, 1977)** $\mathrm{Con}(\mathrm{NF}) \Rightarrow \mathrm{NF} \neq \mathrm{NF}_3$.

Simple Theory of Types TT

- TT is formulated within *multi-sorted* first order logic with countably many sorts

$$X_0, X_1, \cdots.$$

- The language of TT is

$$\{\in_0, \in_1, \cdots\} \cup \{=_0, =_1, \cdots\}.$$

- The atomic formulas are of the form $x^n = y^n$, and $x^n \in_n y^{n+1}$.
- The axioms of TT consist of Extensionality and Comprehension.
- Extensionality: $\big((\forall z^n((z^n \in_n x^{n+1} \leftrightarrow z^n \in_n y^{n+1})\big) \rightarrow x^{n+1} = y^{n+1}$
- Comprehension: $\exists z^{n+1}(\forall y^n(y^n \in_n z^{n+1} \leftrightarrow \varphi(y^n))$.

Specker's discovery

- The ambiguity scheme consists of sentences of the form $\varphi \longleftrightarrow \varphi^+$, where φ^+ is the result of "bumping all types of φ by 1".
- **Theorem (Specker, 1960).** NF *is equiconsient with* TT + Ambiguity.
- **Theorem (Boffa 1988; Kaye-Forster 1991).** NF*is consistent iff there is a model $\mathcal{M}$ of a weak fragment (*KF*) of Zermelo set theory that possess an automorphism j such that for some $m \in \mathcal{M}$, $\mathcal{M} \models |j(m)| = |\mathcal{P}(m)|$.*

Jensen invents NFU

- Quine-Jensen set theory NFU: relax extensionality to allow urelements.
- ZBQC = Zermelo set theory with only Δ_0-Comprehension.
- $\mathrm{NFU}^+ := \mathrm{NFU} + \mathrm{Infinity} + \mathrm{Choice}$.
- $\mathrm{NFU}^- := \mathrm{NFU} +$ "V is finite" $+$ Choice.
- **Theorem (Jensen, 1968)** .
- (1) Con (ZBQC) $\Rightarrow$ Con (NFU^+).
- (2) Con (PA) $\Rightarrow$ Con (NFU^-).
- (3) *If* ZF *has an ω-standard model, then* NFU *has an ω-standard model.*

Boffa's simplification of Jensen's proof

- Arrange a model $\mathcal{M} := (M, E)$ of ZBQC, and an automorphism j of $\mathcal{M}$ such that
- (a) For some infinite $\alpha \in \mathcal{M}$, $j(\alpha) > \alpha$, and
- (b) V_α exists in $\mathcal{M}$;
- Define E_{new} on $V_\alpha^{\mathcal{M}}$ by:
  - $x\ E_{\text{new}}\ y$ iff $j(x)Ey$ and $\mathcal{M} \models y \in V_{j(\alpha)+1}$.

Reversal

- Hinnion and later (independently) Holmes showed that in NFU^+ one can interpret (1) a 'Zermelian structure' Z that satisfies ZFC $\setminus$ $\{$Power Set$\}$, and (2) a nontrivial endomorphism k of Z onto a proper initial segment of Z.
- The endomorphism k can be used to "unravel" Z to a model $\overline{Z}$ of ZBQC that has a nontrivial automorphism j.
- **Theorem (Jensen-Boffa-Hinion).** NFU^+ *has a model iff there is a model* $\mathfrak{M}$ *of* Mac *that has an automorphism* j *such that for some infinite ordinal* α *of* $\mathfrak{M}$,

$$\left(2^{|\alpha|}\right)^{\mathfrak{M}} \leq j(\alpha).$$

Now, about NFU$^-$

- Modulo the work of Jensen (and its simplification by Boffa), in order to arrange a model of NFU$^-$ it suffices to build a model $\mathcal{M}$ of $(I\Delta_0 + \mathrm{Exp})$ with a nontrivial automorphism j such that for some $m \in \mathcal{M}$, $2^m \leq j(m)$.
- Solovay (2002, unpublished) has proved the following:
- $(I\Delta_0 +$ Superexp$) \vdash$

$$\mathrm{Con}(\mathrm{NFU}^-) \iff \mathrm{Con}(I\Delta_0 + \mathrm{Exp}).$$

- $(I\Delta_0 + \mathrm{Exp}) + \mathrm{Con}(I\Delta_0 + \mathrm{Exp}) \nvdash$

$$\mathrm{Con}(\mathrm{NFU}^-).$$

Natural extensions of NFUA^-

- $\mathrm{NFUA}^- := \mathrm{NFU}^- +$ "every Cantorian set is strongly Cantorian".
- $\mathrm{VA} := I\Delta_0 +$ "j is a nontrivial automorphism whose fixed point set is downward closed".
- **Theorem (E, 2006)**. VA *can be (faithfully) interpreted in* NFUA^-.
- **Corollary**. ACA_0 *is (faithfully) interpretable into* VA.
- **Corollary (Solovay-E, 2006.)** *What* NFUA^- *knows about Cantorian arithmetic is precisely* PA.

Automorphisms that move a tail segment

- **Theorem (E, 2006).** *The following two conditions are equivalent for any model $\mathcal{M}$ of the language of arithmetic:*
- (a) $\mathcal{M}$ *satisfies* PA.
- (b) $\mathcal{M} = \mathit{fix}(j)$ *for some nontrivial automorphism j of an end extension $\mathcal{N}$ of $\mathcal{M}$ that satisfies* $I\Delta_0$.

Largest initial segments fixed by automorphisms

- For $j \in \mathrm{Aut}(\mathcal{M})$, $I_{\mathrm{fix}}(j) := \{m \in M : \forall x \leq m\ (j(x) = x)\}$.
- **Theorem (E, 2006).** *The following two conditions are equivalent for a countable model $\mathcal{M}$ of the language of arithmetic:*
- (a) $\mathcal{M} \models I\Delta_0 + B\Sigma_1 + \mathrm{Exp}$.
- (b) $\mathcal{M} = I_{\mathrm{fix}}(j)$ *for some nontrivial automorphism j of an end extension $\mathcal{N}$ of $\mathcal{M}$ that satisfies $I\Delta_0$.*

- Here $B\Sigma_1$ is the Σ_1-collection scheme consisting of the universal closure of formulae of the form

$$[\forall x < a\ \exists y\ \varphi(x, y)] \rightarrow [\exists z\ \forall x < a\ \exists y < z\ \varphi(x, y)],$$

where φ is a Δ_0-formula.

What about NFUA$^+$?

- NFUA$^+$:= NFU$^+$+ "every Cantorian set is strongly Cantorian".
- Λ_0 is

$$\{\text{"there is an } n\text{-Mahlo cardinal"}: \ n \in \omega\}.$$

- **Theorem (Solovay 1995, unpublished)**:

$$\mathrm{Con}(\mathrm{NFUA}^+) \iff \mathrm{Con}(\mathrm{ZFC} + \Lambda_0).$$

What NFUA$^+$ knows about Zermelo-style set theory

- $\Lambda := \{$"there is an n-Mahlo κ with $V_\kappa \prec_n \mathbf{V}$": $n \in \omega\}$.
- Λ_0 is weaker than Λ, but $\mathrm{Con}(\mathrm{ZF} + \Lambda_0) \Longleftrightarrow \mathrm{Con}(\mathrm{ZFC} + \Lambda)$.
- **Theorem (E, 2003)** GBC + "**Ord** is weakly compact" *is (faithfully) interpretable in* NFUA$^+$.
- **Theorem (E, 2003)** *The first order part of* GBC + "**Ord** is weakly compact" *is precisely* Λ.
- **Corollary.** *What* NFUA$^+$ *knows about CZ is precisely* ZFC + Λ.

What is NFUB?

- $\text{NFUB}^{+/-}$ is an extension of $\text{NFUA}^{+/-}$, obtained by adding a scheme that ensures that the intersection of every definable class with CZ is coded by some set.
- Holmes (1998) showed that NFUB^{+} canonically interprets KMC plus "**Ord** is weakly compact".
- Solovay (2000) constructed a model of NFUB^{+} from a model of KMC plus "**Ord** is weakly compact" in which "$\mathbf{V} = \mathbf{L}$".
- In work to appear, I have extended Solovay's construction to show that what NFUB^{+} knows about its canonical Kelley-Morse model is precisely:

  KMC plus "**Ord** is weakly compact" plus Dependent Choice Scheme

On NFUB^-

- Z_2 is second order arithmetic (also known as *analysis*), and *DC* is the scheme of *Dependent Choice*.
- **Theorem (E 2006).** NFUB^- *canonically interprets a model of* $Z_2 + DC$. *Moreover, every countable model of* $Z_2 + DC$ *is isomorphic to the canonical model of analysis of some model of* $NFUB^-$.
- **Corollary.** *What* NFUB^- *knows about Analysis is precisely* $\mathrm{Z}_2 + \mathrm{DC}$.

What NFU$^+$ knows about Cantorian sets

- Let $\mathrm{KP}^{\mathcal{P}}$ be the natural extension of KP in which Σ_1 is replaced by $\Sigma_1^{\mathcal{P}}$.
- For a model $\mathcal{M}$ of $\mathrm{KP}^{\mathcal{P}}$, and an automorphism j of $\mathcal{M}$, let $\mathbf{V}_{\mathrm{fix}}(\mathcal{M}, j)$ be the *longest rank initial segment* fixed by j.
- **Theorem** (E forthcoming) *The following are equivalent for a theory T in the language $\{\in\}$:*
- **(a)** T is a completion of $\mathrm{KP}^{\mathcal{P}}$.
- **(c)** There is a model $\mathcal{M}$ of $\mathrm{NFU}^+ + \mathrm{AxCount}$ such that T is the first order theory of ("the largest rank initial segment of the Cantorian part of **V**")$^{\mathcal{M}}$.