ω-MODELS OF FINITE SET THEORY

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Abstract. Finite set theory, here denoted $\mathsf{ZF}_{\mathsf{fin}}$, is the theory obtained by replacing the axiom of infinity by its negation in the usual axiomatization of ZF (Zermelo-Fraenkel set theory). An *ω-model* of $\mathsf{ZF}_{\mathsf{fin}}$ is a model in which every set has at most finitely many elements (as viewed externally). Mancini and Zambella (2001) employed the Bernays-Rieger method of permutations to construct a recursive ω-model of $\mathsf{ZF}_{\mathsf{fin}}$ that is nonstandard (i.e., not isomorphic to the hereditarily finite sets V_ω). In this paper we initiate the metamathematical investigation of ω-models of $\mathsf{ZF}_{\mathsf{fin}}$. In particular, we present a new method for building ω-models of $\mathsf{ZF}_{\mathsf{fin}}$ that leads to a perspicuous construction of recursive nonstandard ω-models of $\mathsf{ZF}_{\mathsf{fin}}$ without the use of permutations. Furthermore, we show that every recursive model of $\mathsf{ZF}_{\mathsf{fin}}$ is an ω-model. The central theorem of the paper is the following:

Theorem A. *For every graph (A, F), where F is a set of unordered pairs of A, there is an ω-model $\mathfrak{M}$ of $\mathsf{ZF}_{\mathsf{fin}}$ whose universe contains A and which satisfies the following conditions:*

(1) *(A, F) is definable in $\mathfrak{M}$;*
(2) *Every element of $\mathfrak{M}$ is definable in $(\mathfrak{M}, a)_{a\in A}$;*
(3) *If (A, F) is pointwise definable, then so is $\mathfrak{M}$;*
(4) $\mathsf{Aut}(\mathfrak{M}) \cong \mathsf{Aut}(A, F)$.

Theorem A enables us to build a variety of ω-models with special features, in particular:

Corollary 1. *Every group can be realized as the automorphism group of an ω-model of $\mathsf{ZF}_{\mathsf{fin}}$.*

Corollary 2. *For each infinite cardinal κ there are 2^κ rigid nonisomorphic ω-models of $\mathsf{ZF}_{\mathsf{fin}}$ of cardinality κ.*

Corollary 3. *There are continuum-many nonisomorphic pointwise definable ω-models of $\mathsf{ZF}_{\mathsf{fin}}$.*

We also establish that PA (Peano arithmetic) and $\mathsf{ZF}_{\mathsf{fin}}$ are not bi-interpretable by showing that they differ even for a much coarser notion of equivalence, to wit *sentential equivalence*.

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1. INTRODUCTION

In 1953, Kreisel [Kr] and Mostowski [Mos] independently showed that certain finitely axiomatizable systems of set theory formulated in an *expansion* of the usual language $\{\in\}$ of set theory do not possess any recursive models. This result was improved in 1958 by Rabin [Ra] who found a "familiar" finitely axiomatizable first order theory that has no recursive model: Gödel-Bernays[1] set theory GB without the axiom of infinity (note that GB can be formulated in the language $\{\in\}$ with no extra symbols). These discoveries were overshadowed by Tennenbaum's celebrated 1961 theorem that characterizes the standard model of PA (Peano arithmetic) as the only recursive model of PA up to isomorphism, thereby shifting the focus of the investigation of the complexity of models from set theory to arithmetic. We have come a long way since Tennenbaum's pioneering work in understanding the contours of the "Tennenbaum boundary" that separates those fragments of PA that have a recursive nonstandard model (such as $I\mathsf{Open}$) from those which do not (such as $I\exists_1$), but the study of the complexity of models of arithmetic and its fragments remains a vibrant research area with many intriguing open questions.[2]

The point of departure for the work presented here is Mancini and Zambella's 2001 paper [MZ] that focuses on *Tennenbaum phenomena in set theory*. Mancini and Zambella introduced a weak fragment (dubbed $\mathsf{KP}\Sigma_1$ [3]) of Kripke-Platek set theory KP, and showed that the only recursive model of $\mathsf{KP}\Sigma_1$ up to isomorphism is the standard one, i.e., $(V_\omega, \in)$, where V_ω is the set of hereditarily finite sets. In contrast, they used the *Bernays-Rieger*[4] permutation method to show that the theory

[1]We have followed Mostowski's lead in our adoption of the appellation GB, but some set theory texts refer to this theory as BG. To make matters more confusing, the same theory is also known in the literature as VNB (*von Neumann-Bernays*) and NBG (*von Neumann-Bernays-Gödel*).

[2]See, e.g., the papers by Kaye [Ka] and Schmerl [Sch-2] in this volume, and Mohsenipour's [Moh].

[3]The axioms of $\mathsf{KP}\Sigma_1$ consist of Extensionalty, Pairs, Union, Foundation, Δ_0-Comprehension, Δ_0-Collection, and the scheme of $\in$-Induction (defined in part (f) of Remark 2.2) only for Σ_1 formulas . Note that $\mathsf{KP}\Sigma_1$ does not include the axiom of infinity.

[4]In [MZ] this method is incorrectly referred to as the *Fraenkel-Mostowski* permutation method, but this method was invented by Bernays (announced in [Be-1, p. 9], and presented in [Be-2]) and fine-tuned by Rieger [Ri] in order to build models of set theory that violate the regularity (foundation) axiom, e.g., by containing

$\mathsf{ZF_{fin}}$ obtained by replacing the axiom of infinity by its negation in the usual axiomatization of ZF (Zermelo-Fraenkel set theory) has a recursive nonstandard model. The Mancini-Zambella recursive nonstandard model also has the curious feature of being an *ω-model* in the sense that every element of the model, as viewed externally, has at most finitely many members, a feature that caught our imagination and prompted us to initiate the systematic investigation of the metamathematics of ω-models of $\mathsf{ZF_{fin}}$.

The plan of the paper is as follows. Preliminaries are dealt with in Section 2, in which we review key definitions, establish notation, and discuss a host of background results. In Section 3 we present a simple robust construction of ω-models of $\mathsf{ZF_{fin}}$ (Theorem 3.4), a construction that, among other things, leads to a perspicuous proof of the existence of ω-models of $\mathsf{ZF_{fin}}$ in every infinite cardinality (Corollary 3.7), and the existence of infinitely many nonisomorphic recursive nonstandard ω-models of $\mathsf{ZF_{fin}}$ without the use of permutations (Corollary 3.9, Remark 3.10(b)). In the same section, we also show that $\mathsf{ZF_{fin}}$ is not completely immune to Tennenbaum phenomena by demonstrating that every recursive model of $\mathsf{ZF_{fin}}$ is an ω-model (Theorem 3.11). In Section 4 we fine-tune the method of Section 3 to show the existence of a wealth of nonisomorphic ω-models of $\mathsf{ZF_{fin}}$ with special features. The central theorem of Section 4 is Theorem 4.2 which shows that every graph can be canonically coded into an ω-model of $\mathsf{ZF_{fin}}$. Coupled with classical results in graph theory, this result yields many corollaries. For example, every group (of any cardinality) can be realized as the automorphism group of an ω-model of $\mathsf{ZF_{fin}}$ (Corollary 4.4), and there are continuum-many nonisomorphic pointwise definable ω-models of $\mathsf{ZF_{fin}}$ (Corollary 4.7). In Section 5 we establish that PA (Peano arithmetic) and $\mathsf{ZF_{fin}}$ are not bi-interpretable by showing that they differ even for a much coarser notion of equivalence, to wit *sentential equivalence* (Theorem 5.1). This complements the work of Kaye-Wong[5] [KW] on the definitional equivalence (or synonymy) of PA and $\mathsf{ZF_{fin}}$ + "every set has a transitive closure". We close the paper with Section 6, in which we present open questions and concluding remarks.

We also wish to take this opportunity to thank the anonymous referee for helpful comments and corrections, and Robert Solovay for catching a blooper in an earlier draft of this paper.

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sets x such that $x = \{x\}$. In contrast, the Fraenkel-Mostowski method is used to construct models of set theory with atoms in which the axiom of choice fails.

[5]See Remark 2.2(f) for more detail concerning the Kaye-Wong paper.

2. PRELIMINARIES

In this section we recall some key definitions, establish notation, and review known results.

Definitions/Observations 2.1.

(a) Models of set theory are *directed graphs*[6] (hereafter: *digraphs*), i.e., structures of the form $\mathfrak{M} = (M, E)$, where E is a binary relation on M that interprets $\in$. We often write xEy as a shorthand for $\langle x, y\rangle \in E$. For $c \in M$, c_E is the set of "elements" of c, i.e.,

$$c_E := \{m \in M : mEc\}.$$

$\mathfrak{M}$ is *nonstandard* if E is not well-founded, i.e., if there is a sequence $\langle c_n : n \in \omega\rangle$ of elements of M such that $c_{n+1}Ec_n$ for all $n \in \omega$.

(b) We adopt the terminology of Baratella and Ferro [BF] of using EST (elementary set theory) to refer to the following theory of sets

$$\mathsf{EST} := \mathsf{Extensionality} + \mathsf{Empty\ Set} + \mathsf{Pairs} + \mathsf{Union} + \mathsf{Replacement}.$$

(c) The theory $\mathsf{ZF_{fin}}$ is obtained by replacing the axiom of infinity by its negation in the usual axiomatization of ZF (Zermelo-Fraenkel set theory). More explicitly:

$$\mathsf{ZF_{fin}} := \mathsf{EST} + \mathsf{Power\ set} + \mathsf{Regularity}^7 + \neg\mathsf{Infinity}.$$

Here $\mathsf{Infinity}$ is the usual axiom of infinity, i.e.,

$$\mathsf{Infinity} := \exists x\ \left(\emptyset \in x \wedge \forall y\,(y \in x \to y^+ \in x)\right),$$

where $y^+ := y \cup \{y\}$.

(d) $\mathsf{Tran}(x)$ is the first order formula that expresses the statement "x is transitive", i.e.,

$$\mathsf{Tran}(x) := \forall y \forall z\,(z \in y \in x \to z \in x).$$

(e) $\mathsf{TC}(x)$ is the first order formula that expresses the statement "the transitive closure of x is a set", i.e.,

$$\mathsf{TC}(x) := \exists y\,(x \subseteq y \wedge \mathsf{Tran}(y)).$$

Overtly, the above formula just says that some superset of x is transitive, but it is easy to see that $\mathsf{TC}(x)$ is equivalent within EST to the

[6]We will also have ample occasion to deal with (undirected) graphs, i.e., structures of the form (A, F), where F is a set of *unordered* pairs from A.

[7]The regularity axiom is also known as the foundation axiom, stating that every nonempty set has an $\in$-minimal element.

following statement expressing "there is a smallest transitive set that contains x"

$$\exists y\,(x \subseteq y \wedge \mathsf{Tran}(y) \wedge \forall z\,((x \subseteq z \wedge \mathsf{Tran}(z)) \rightarrow y \subseteq z)).$$

(f) TC denotes the *transitive closure* axiom

$$\mathsf{TC} := \forall x\ \mathsf{TC}(x).$$

Let V_ω be the set of hereditarily finite sets. It is easy to see that $\mathsf{ZF_{fin}} + \mathsf{TC}$ holds in V_ω. However, it has long been known[8] that $\mathsf{ZF_{fin}} \nvdash \mathsf{TC}$.

(g) $\mathbb{N}(x)$ [read as "x is a natural number"] is the formula

$$\mathsf{Ord}(x) \wedge \forall y{\in}x^+\,(y \neq \emptyset \rightarrow \exists z\,(\mathsf{Ord}(z) \wedge y = z^+)),$$

where $\mathsf{Ord}(x)$ expresses "x is a (von Neumann) ordinal", i.e., "x is a transitive set that is well-ordered by $\in$". It is well-known that with this interpretation, the full induction scheme $\mathsf{Ind}_{\mathbb{N}}$, consisting of the universal closure of formulas of the following form is provable within EST:

$$\left(\theta(0) \wedge \forall x\left(\mathbb{N}(x) \wedge \theta(x) \rightarrow \theta(x^+)\right)\right) \rightarrow \forall x\left(\mathbb{N}(x) \rightarrow \theta(x)\right).$$

Note that θ is allowed to have suppressed parameters, and these parameters are not required to lie in $\mathbb{N}$. Coupled with the fact that $\mathsf{ZF_{fin}}$ is a sequential theory[9], this shows that for each positive integer n, there is a formula $\mathsf{Tr}_n(x)$ such that, provably in $\mathsf{ZF_{fin}}$, $\mathsf{Tr}_n(x)$ is a truth-predicate for the class of formulas Q_n with at most n alternations of quantifiers. Coupled with the fact that any formula in Q_n that is provable in predicate logic has a proof all of whose components lie in Q_n (which follows from Herbrand's Theorem [HP, Thm 3.30, Ch.III]), this shows that $\mathsf{ZF_{fin}}$ is *essentially reflexive*, i.e., any consistent extension of $\mathsf{ZF_{fin}}$ proves the consistency of each of its finite subtheories. Therefore $\mathsf{ZF_{fin}}$ is not finitely axiomatizable.

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[8]Hájek-Vopěnka [HV] showed that TC is not provable in the theory $\mathsf{GB_{fin}}$, which is obtained from GB (Gödel-Bernays theory of classes) by replacing the axiom of infinity by its negation. Later Hauschild [Ha] gave a direct construction of a model of $\mathsf{ZF_{fin}} + \neg\mathsf{TC}$.

[9]Sequential theories are those that are equipped with a "β-function" for coding sequences. More specifically, using $\langle x, y\rangle$ for the usual Kuratowski ordered pair of x and y, the function $\beta(x, y)$ defined via $\beta(x, y) = z$ iff $\langle x, z\rangle \in y$, conveniently serves in EST as a β-function (when y is restricted to "functional" sets, i.e., y should contain at most one ordered pair $\langle x, z\rangle$ for a given x).

(h) For a model $\mathfrak{M} \models \mathsf{EST}$, and $x \in M$, we say that x is $\mathbb{N}$-*finite* if there is a bijection in $\mathfrak{M}$ between x and some element of $\mathbb{N}^{\mathfrak{M}}$. Let:

$$(V_\omega)^{\mathfrak{M}} := \{m \in M : \mathfrak{M} \models \text{“}\mathsf{TC}(m) \text{ and } m \text{ is } \mathbb{N}\text{-finite”}\}$$

It is easy to see that

$$(V_\omega)^{\mathfrak{M}} \models \mathsf{ZF}_{\mathsf{fin}} + \mathsf{TC}.$$

This provides an interpretation of the theory $\mathsf{ZF}_{\mathsf{fin}} + \mathsf{TC}$ within EST. On the other hand, the existence of recursive nonstandard models of $\mathsf{ZF}_{\mathsf{fin}}$ shows that, conversely, $\mathsf{ZF}_{\mathsf{fin}} + \neg\mathsf{TC}$ is also interpretable in EST. One can show that the above interpretation of $\mathsf{ZF}_{\mathsf{fin}} + \mathsf{TC}$ within EST is *faithful* (i.e., the sentences that are provably true in the interpretation are precisely the logical consequences of $\mathsf{ZF}_{\mathsf{fin}} + \mathsf{TC}$), but that the Mancini-Zambella interpretation [MZ, Theorem 3.1] of $\mathsf{ZF}_{\mathsf{fin}} + \neg\mathsf{TC}$ within EST, is not faithful[10].

(i) $\tau(n, x)$ is the term expressing “the n-th approximation to the transitive closure of $\{x\}$ (where n is a natural number)”. Informally speaking,

- $\tau(0, x) = \{x\}$;
- $\tau(n + 1, x) = \tau(n, x) \cup \{y : \exists z\, (y \in z \in \tau(n, x)\}$.

Thanks to the coding apparatus of EST for dealing with finite sequences, and the provability of $\mathsf{Ind}_{\mathbb{N}}$ within EST (both mentioned earlier in part (g)), the above informal recursion can be formalized within EST to show that

$$\mathsf{EST} \vdash \forall n \forall x\, (\mathbb{N}(n) \to \exists! y\, (\tau(n, x) = y)).$$

This leads to the following important observation:

(j) Even though the transitive closure of a set need not form a set in EST (or even in $\mathsf{ZF}_{\mathsf{fin}}$), for an ω-model $\mathfrak{M}$ the transitive closure $\tau(c)$ of $\{c\}$ is *first order definable* via:

$$\tau^{\mathfrak{M}}(c) := \{m \in M : \mathfrak{M} \models \exists n\, (\mathbb{N}(n) \wedge m \in \tau(n, c))\}.$$

This shows that, in the worst case scenario, transitive closures behave like proper classes in ω-models of $\mathfrak{M}$. Note that if the set $\mathbb{N}^{\mathfrak{M}}$ of natural numbers of $\mathfrak{M}$ contains nonstandard elements, then the *external* transitive closure

$$\bigcup_{n \in \omega} \tau^{\mathfrak{M}}(n, c)$$

[10]Because the interpretation provably satisfies the sentence “the universe can be built from the transitive closure of an element whose transitive closure forms an ω^*-chain”, a sentence that is not a theorem of $\mathsf{ZF}_{\mathsf{fin}} + \neg\mathsf{TC}$.

might be a proper subset of the $\mathfrak{M}$-transitive closure $\tau^{\mathfrak{M}}(c)$. However, if $\mathfrak{M}$ is an ω-model, then the $\mathfrak{M}$-transitive closure of $\{c\}$ coincides with the external transitive closure of $\{c\}$.

Remark 2.2.

(a) Vopěnka [Vo-2] has shown that $\mathsf{ZF_{fin}} \setminus \{\mathsf{Regularity}\}$ is provable from the fragment (dubbed VF in [BF]) of AST (Alternative Set Theory), whose axioms consist of $\mathsf{Extensionality}$, $\mathsf{Empty\ set}$, $\mathsf{Adjunction}$ (given sets x and y, we have that $x \cup \{y\}$ exists), and the schema of $\mathsf{Set\text{-}induction}$, consisting of the universal closure of formulas of the form (θ is allowed to have suppressed parameters)

$$(\theta(0) \wedge \forall x \forall y\ (\theta(x) \rightarrow \theta(x \cup \{y\}))) \rightarrow \forall x\, \theta(x).$$

(b) Besides $\neg\mathsf{Infinity}$ there are at least two other noteworthy first order statements that can be used to express "every set is finite":

- $\mathsf{Fin}_{\mathbb{N}}$: Every set is $\mathbb{N}$-finite.
- Fin_{D} : Every set is Dedekind-finite, i.e., no set is equinumerous to a proper subset of itself.

It is easy to see that EST proves $\mathsf{Fin}_{\mathbb{N}} \rightarrow \mathsf{Fin}_{D} \rightarrow \neg\,\mathsf{Infinity}$. By a theorem of Vopěnka [Vo-1], $\mathsf{Power\ set}$ and the well-ordering theorem (and therefore the axiom of choice) are provable within $\mathsf{EST} + \mathsf{Fin}_{\mathbb{N}}$ (see [BF, Theorem 5] for an exposition).

(c) Kunen [BF, Sec. 7] has shown the consistency of the theory $\mathsf{EST} + \neg\mathsf{Infinity} + \neg\mathsf{Fin}_{\mathbb{N}}$ using the Fraenkel-Mostowski permutations method.

(d) In contrast with Kunen's aforementioned result, $\mathsf{Fin}_{\mathbb{N}}$ is provable within $\mathsf{ZF_{fin}} \setminus \{\mathsf{Regularity}\}$ (i.e., within $\mathsf{EST} + \mathsf{Powerset} + \neg\,\mathsf{Infinity}$). To see this, work in $\mathsf{ZF_{fin}} \setminus \{\mathsf{Regularity}\}$ and suppose to the contrary that there is an element x that is not $\mathbb{N}$-finite. By an easy induction, we find that for every natural number n there is a subset y of x that is equinumerous with n. Now we define the function F on the powerset of x by:

$$F(y) := n, \text{ if } n \text{ is a natural number that is equinumerous with } y;$$
$$\text{otherwise } F(y) = 0.$$

It is easy to see that the range of F is ω. Hence by $\mathsf{Replacement}$, ω is a set. Quod non.

(e) As mentioned earlier in part (a), $\mathsf{VF} \vdash \mathsf{ZF_{fin}} \setminus \{\mathsf{Regularity}\}$. The provability of both $\mathsf{Fin}_{\mathbb{N}}$ and $\mathsf{Ind}_{\mathbb{N}}$ in $\mathsf{ZF_{fin}}$ can be used to show that $\mathsf{VF} + \mathsf{Regularity}$ and $\mathsf{EST} + \mathsf{Fin}_{\mathbb{N}} + \mathsf{Regularity}$ axiomatize the same first order theory as $\mathsf{ZF_{fin}}$.

(f) As observed by Kaye and Wong [KW, Prop.12] within EST, the principle TC is equivalent to the scheme of $\in$-$\mathsf{Induction}$ consisting of statements of the following form (θ is allowed to have suppressed parameters)

$$\forall y\,(\forall x{\in}y\ \theta\,(x) \rightarrow \theta(y)) \rightarrow \forall z\,\theta(z).$$

In the same paper Kaye and Wong showed the following strong form of bi-interpretability[11] between PA and $\mathsf{ZF}_{\mathsf{fin}} + \mathsf{TC}$, known as *definitional equivalence* (or *synonymity,* in the sense of [Vi-2, Sec.4.8.2]) by showing that:

(1) TC holds in the Ackermann interpretation Ack of $\mathsf{ZF}_{\mathsf{fin}}$ within PA, i.e., $\mathsf{Ack} : \mathsf{ZF}_{\mathsf{fin}} + \mathsf{TC} \rightarrow \mathsf{PA}$; and

(2) There is an interpretation $\mathsf{B} : \mathsf{PA} \rightarrow \mathsf{ZF}_{\mathsf{fin}} + \mathsf{TC}$ such that $\mathsf{Ack} \circ \mathsf{B} = \mathsf{id}_{\mathsf{PA}}$ and $\mathsf{B} \circ \mathsf{Ack} = \mathsf{id}_{\mathsf{ZF}_{\mathsf{fin}}+\mathsf{TC}}$.

The above result suggests that, contrary to a popular misconception, PA might *not* be bi-interpretable with $\mathsf{ZF}_{\mathsf{fin}}$ alone. Indeed, we shall establish a strong form of the failure of the bi-interpretability between PA and $\mathsf{ZF}_{\mathsf{fin}}$ in Theorem 5.1. Note that, in contrast, by a very general result[12] in interpretability theory, PA and $\mathsf{ZF}_{\mathsf{fin}}$ are mutually faithfully interpretable.

3. BUILDING ω-MODELS

Definition 3.1. Suppose $\mathfrak{M}$ is a model of EST. $\mathfrak{M}$ is an *ω-model* if $|x_E|$ is finite for every $x \in M$ satisfying $\mathfrak{M} \models$ "x is $\mathbb{N}$-finite".

It is easy to see that $\mathfrak{M}$ is an ω-model iff $(\mathbb{N}, \in)^{\mathfrak{M}}$ is isomorphic to the standard natural numbers.[13] This observation can be used to show that ω-models of $\mathsf{ZF}_{\mathsf{fin}}$ are precisely the models of the second order theory $\mathsf{ZF}^2_{\mathsf{fin}}$ obtained from $\mathsf{ZF}_{\mathsf{fin}}$ by replacing the replacement scheme by its second order analogue.

[11] See the paragraph preceding Theorem 3.12 for more detail on bi-interpretability.

[12] See [Vi-1, Lemma 5.4] for the precise formulation of this result in a general setting, which shows that any Σ^0_1-sound theory can be faithfully interpreted in a sufficiently strong theory. Visser's result extends earlier work of Lindström [Li, Ch. 6, Sec. 2, Thm 13] which dealt with a similar phenomenon in the specific confines of theories extending PA.

[13] Note that there are really two salient notions of ω-model, to wit the notion we defined here could be called $\omega_{\mathbb{N}}$-model, and the notion defined using 'Dedekind-finite' instead of '$\mathbb{N}$-finite' may be called ω_D-model. For our study of $\mathsf{ZF}_{\mathsf{fin}}$ the choice is immaterial, since $\mathsf{ZF}_{\mathsf{fin}}$ proves that every set is $\mathbb{N}$-finite (see part (d) of Remark 2.2).

The following proposition provides a useful *graph-theoretic* characterization of *ω-models of* $\mathsf{ZF_{fin}}$. Note that even though $\mathsf{ZF_{fin}}$ is not finitely axiomatizable,[14] the equivalence of (a) and (b) of Proposition 3.2 shows that there is a single sentence in the language of set theory whose ω-models are precisely ω-models of $\mathsf{ZF_{fin}}$.

- Recall that a vertex x of a digraph $G := (X, E)$ has *finite in-degree* if x_E is finite; and G is *acyclic* if there is no finite sequence $x_1 \ E \ x_2 \ \cdots \ E \ x_{n-1} \ E \ x_n$ in G with $x_1 = x_n$.

Proposition 3.2. *The following three conditions are equivalent for a digraph $G := (X, E)$:*

(a) *G is an ω-model of* $\mathsf{ZF_{fin}}$.

(b) *G is an ω-model of* Extensionality*,* Empty set*,* Regularity*,* Adjunction *and* $\neg$ Infinity.

(c) *G satisfies the following four conditions:*
(*i*) *E is extensional;*
(*ii*) *Every vertex of G has finite in-degree*;
(*iii*) *G is acyclic; and*
(*iv*) *G has an element of in-degree 0, and for all positive $n \in \omega$,*

$$(X, E) \models \forall x_1 \cdots \forall x_n \ \exists y \ \forall z \ (zEy \leftrightarrow \bigvee_{i=1}^{n} z = x_i).$$

Proof:
(a) $\Rightarrow$ (b): Trivial.

(b) $\Rightarrow$ (c): Assuming (b), (*i*) and (*ii*) are trivially true. (*iv*) is an easy consequence of Empty set and n-applications of Adjunction. To verify (*iii*), suppose to the contrary that $x_1 \ E \ x_2 \cdots \ Ex_{n-1} \ E \ x_n$ is a cycle in G with $x_1 = x_n$. By (*iv*), there is an element $y \in X$ with $y_E = \{x_i : 1 \leq i \leq n\}$. This contradicts Regularity since such a y has no minimal "element".

(c) $\Rightarrow$ (a): Routine, but we briefly comment on the verification of Regularity, which is accomplished by contradiction: if x is a nonempty set with no minimal element, then there exists an external infinite sequence $\langle x_n : n \in \omega \rangle$ of elements of c such that $x_{n+1} \ E \ x_n$ for all $n \in \omega$. Invoking statement (*ii*) of (c), this shows that there are x_m and x_n with $m < n$ such that $x_m = x_n$, which contradicts condition (*iii*) of (c). $\square$

[14] See 2.1(g)

The following list of definitions prepares the way for the first key theorem of this section (Theorem 3.4), which will enable us to build plenty of ω-models of $\mathsf{ZF}_{\mathsf{fin}}$.

Definition 3.3. Suppose $G := (X, E)$ is an extensional, acyclic digraph, all of whose vertices have finite in-degree.

(a) A subset S of X is said to be *coded* in G if there is some $x \in X$ such that $S = x_E$.

(b) $D(G) := \{S \subseteq X : S \text{ is finite and } S \text{ is not coded in } G\}$. We shall refer to $D(G)$ as the *deficiency set of* G.

(c) The infinite sequence of digraphs

$$\langle \mathbb{V}_n(G) : n \in \omega \rangle \,,$$

where $\mathbb{V}_n(G) := (V_n(G), E_n(G))$, is built recursively using the following clauses[15]:

- $V_0(G) := X; \quad E_0(G) := E;$
- $V_{n+1}(G) := V_n(G) \cup D(\mathbb{V}_n(G));$
- $E_{n+1}(G) := E_n(G) \cup \{\langle x, S \rangle \in V_n(G) \times D(\mathbb{V}_n(G)) : x \in S\}.$

(d) $\mathbb{V}_\omega(G) := (V_\omega(G),\ E_\omega(G))$, where

$$V_\omega(G) := \bigcup_{n\in\omega} V_n(G), \quad E_\omega(G) := \bigcup_{n\in\omega} E_n(G).$$

Theorem 3.4. *If $G := (X, E)$ is an extensional, acyclic digraph, all of whose vertices have finite in-degree, then $\mathbb{V}_\omega(G)$ is an ω-model of $\mathsf{ZF}_{\mathsf{fin}}$.*

Proof: We shall show that $\mathbb{V}_\omega(G)$ satisfies the four conditions of (c) of Proposition 3.2. Before doing so, let us make an observation that is helpful for the proof, whose verification is left to the reader (footnote 15 comes handy here).

Observation 3.4.1. *For each $n \in \omega$, $\mathbb{V}_{n+1}(G)$ "end extends" $\mathbb{V}_n(G)$, i.e., if $aE_{n+1}\, b$ and $b \in V_n(G)$, then $a \in V_n(G)$.*

It is easy to see that extensionality is preserved in the passage from $\mathbb{V}_n(G)$ to $\mathbb{V}_{n+1}(G)$. By the above observation, at no point in the construction of $\mathbb{V}_\omega(G)$ a new member is added to an old member, which shows that extensionality is preserved in the limit. It is also easy to

[15] Since we want the elements of X to behave like *urelements*, something could go wrong with this definition if some vertex happens to be a finite set of vertices, or a finite set of finite sets of vertices, etc. A simple way to get the desired effect is to replace X with $X^* = \{\{\{x\}, X\} : x \in X\}$. Then $X^* \cap D(V_n(G)) = \emptyset$ holds for all $n \in \omega$, and all digraphs G with vertex-set X^*.

see that every vertex of $\mathbb{V}_\omega(G)$ has finite in-degree. To verify that $\mathbb{V}_\omega(G)$ is acyclic it suffices to check that each finite approximation $\mathbb{V}_n(G)$ is acyclic, so we shall verify that it is impossible for $\mathbb{V}_{n+1}(G)$ to have a cycle and for $\mathbb{V}_n(G)$ to be acyclic. So suppose there is a cycle $\langle s_i : 1 \leq i \leq k\rangle$ in $\mathbb{V}_{n+1}(G)$, and $\mathbb{V}_n(G)$ is acyclic. Then by Observation 3.4.1, for some i,

$$s_i \in V_{n+1}(G) \setminus V_n(G).$$

This implies that s_i is a member of the deficiency set of $\mathbb{V}_n(G)$, and so $s_i \subseteq V_n(G)$. But for $j = i+1 \pmod k$, we have $s_i\ E_{n+1}\ s_j$. This contradicts the definition of E_{n+1}, thereby completing our verification that G is acyclic. $\square$

Remark 3.5.

(a) The proof of Theorem 3.4 makes it clear that G is end extended by $\mathbb{V}_\omega(G)$; and every element of $\mathbb{V}_\omega(G)$ is first order definable in the structure $(\mathbb{V}_\omega(G), c)_{c\in X}$.

(b) Recall from part (i) of Definition 2.1 that $\tau(n,c)$ denotes the n-th approximation to the transitive closure $\tau(c)$ of $\{c\}$. For any $c \in V_\omega(G)$, a tail of $\tau(n,c)$ lies in G, i.e., $\tau(c)\backslash\tau(n,c) \subseteq G$ for sufficiently large n.

Example 3.6.

(a) For every transitive $S \subseteq V_\omega$, $\mathbb{V}_\omega(S, \in) \cong (V_\omega, \in)$.

(b) Let $G_\omega := (\omega, \{\langle n+1, n\rangle : n \in \omega\})$. *$\mathbb{V}_\omega(G_\omega)$ is our first concrete example of a nonstandard ω-model of* $\mathsf{ZF_{fin}}$.

Corollary 3.7. $\mathsf{ZF_{fin}}$ *has ω-models in every infinite cardinality.*

Proof: For any (finite or infinite) set I, and any digraph $G = (X, E)$, let $I \times G := (I \times X, F)$, where

$$\langle i, x\rangle\ F\ \langle j, y\rangle \Leftrightarrow i = j \wedge xEy.$$

Note that $I \times G$ is the disjoint union of $|I|$ copies of G. It is easy to see that if G is an extensional, acyclic digraph, all of whose vertices have finite in-degree, and G has no vertex v with $v_E = \emptyset$, then $I \times G$ shares the same features. Therefore if G_ω is as in Example 3.6(b), and I is infinite, then $\mathbb{V}_\omega(I \times G_\omega)$ is an ω-model of $\mathsf{ZF_{fin}}$ of the same cardinality as I. $\square$

Remark 3.8. Corollary 3.7 shows that in contrast with $\mathsf{ZF_{fin}} + \mathsf{TC}$, within $\mathsf{ZF_{fin}}$ there is no definable bijection between the universe and the set of natural numbers. Furthermore, since $\mathbb{V}_\omega(\{0,1\} \times G_\omega)$ has an automorphism of order 2, there is not even a definable linear ordering of the universe available in $\mathsf{ZF_{fin}}$.

We need to introduce a key definition before stating the next result:

- A digraph $G = (\omega, E)$ is said to be *highly recursive* if (1) for each $n \in \omega$, n_E is finite; (2) The map $n \mapsto c(n_E)$ is recursive, where c is a canonical code[16] for n_E; and (3) $\{c(n_E) : n \in \omega\}$ is recursive.

Corollary 3.9. $\mathsf{ZF}_{\mathsf{fin}}$ *has nonstandard highly recursive ω-models.*

Proof: The constructive nature of the proof of Theorem 3.4 makes it evident that if $G = (\omega, E)$ is highly recursive digraph, then there is highly recursive $R \subseteq \omega^2$ such that $(\omega, R) \cong \mathbb{V}_\omega(G)$. Since the digraph G_ω of Example 3.6(b) is easily seen to be highly recursive, we may conclude that there is a highly recursive $F \subseteq \omega^2$ such that $(\omega, F) \cong \mathbb{V}_\omega(G_\omega)$. $\square$

Remark 3.10.

(a) Generally speaking, if $\mathfrak{M}$ is a highly recursive ω-model of $\mathsf{ZF}_{\mathsf{fin}}$, then the set of natural numbers $\mathbb{N}^{\mathfrak{M}}$ is also recursive. However, it is easy to construct a recursive ω-model of $\mathsf{ZF}_{\mathsf{fin}}$ in which the set of natural numbers $\mathbb{N}^{\mathfrak{M}}$ is not recursive. With more effort, one can even build a recursive ω-model of $\mathsf{ZF}_{\mathsf{fin}}$ that is not isomorphic to a recursive model with a recursive set of natural numbers.

(b) A minor modification of the proof of Corollary 3.9 shows that there are infinitely many pairwise elementarily inequivalent highly recursive models of $\mathsf{ZF}_{\mathsf{fin}}$. This is based on the observation that each digraph in the family $\{I \times G_\omega : |I| \in \omega\}$ has a highly recursive copy, and for each positive $n \in \omega$ the sentence

$$\exists x_1 \cdots \exists x_n \; \Big(\bigwedge_{1 \leq i \leq n} \text{“}\tau(x_i) \text{ is infinite”} \wedge \bigwedge_{1 \leq i < j \leq n} \tau(x_i) \cap \tau(x_j) = \emptyset \Big)$$

holds in $\mathbb{V}_\omega(I \times G_\omega)$ iff $|I| \geq n$. In particular, this shows that in contrast to PA and $\mathsf{ZF}_{\mathsf{fin}} + \mathsf{TC}$, $\mathsf{ZF}_{\mathsf{fin}} + \neg\mathsf{TC}$ has infinitely many nonisomorphic recursive models. However, as shown by the next theorem, $\mathsf{ZF}_{\mathsf{fin}}$ does not entirely escape the reach of Tennenbaum phenomena.[17]

Theorem 3.11. *Every recursive model of* $\mathsf{ZF}_{\mathsf{fin}}$ *is an ω-model.*

Proof: The theory $\mathsf{ZF}_{\mathsf{fin}}$ is existentially rich (see [Sch-2, Definition 1.1]) as is shown by the recursive sequence $\langle \theta_n(x) : n \in \omega \rangle$, where $\theta_n(x)$ is

[16] For example, c can be defined via $c(X) = \sum_{n \in X} 2^n$.

[17] The proof of Theorem 3.11 indeed shows that every recursive model of EST is an ω-model.

the existential formula

$$\exists x_0, x_1, \ldots, x_{n+2}, y_0, y_1, \ldots, y_{n+2} \Big(\bigwedge_{i<j\leq n+2} x_i \neq x_j \\ \wedge \bigwedge_{i\leq n+2} (x_i \in y_i \wedge x_{i+1} \in y_i \in x) \Big).$$

Here, we understand x_{n+3} to be x_0. To see that the $\theta_n(x)$ are existentially rich, let $i_0, i_1, i_2, \cdots$ be sufficiently fast growing (letting $i_n = \frac{n^2+5n}{2}$ is good enough) and then let

$$E_n = \{\{i, i+1\} : i_n \leq i < i_n + n - 1\} \cup \{i_n + n - 1, i_n\}.$$

If I is a finite subset of ω and $a = \bigcup_{n\in I} E_n$, then

$$\mathsf{ZF}_{\mathsf{fin}} \vdash \theta_n(a) \text{ if } n \in I, \text{ and } \mathsf{ZF}_{\mathsf{fin}} \vdash \neg\theta_n(a) \text{ if } n \notin I.$$

Moreover, thanks to the coding apparatus available in $\mathsf{ZF}_{\mathsf{fin}}$, there is a binary formula $\theta(n, x)$ such that for each $n \in \omega$,

$$(*) \qquad \mathsf{ZF}_{\mathsf{fin}} \vdash \forall x\, (\theta(n, x) \leftrightarrow \theta_n(x))\,.$$

Suppose $\mathfrak{M}$ is a non ω-model of $\mathsf{ZF}_{\mathsf{fin}}$, and let A, B be two disjoint, recursively inseparable, recursively enumerable sets. Since $\mathfrak{M}$ satisfies $\mathsf{Ind}_{\mathbb{N}}$, we can use Overspill to arrange a definable subset S of $\mathfrak{M}$ such that $A \subseteq S$ and B is disjoint from S. If $S(x)$ is a defining formula for S in $\mathfrak{M}$, then for each $n \in \omega$, there is c in $\mathfrak{M}$ such that

$$\mathfrak{M} \models \forall i < n\, (\theta_i(c) \leftrightarrow S(i))\,.$$

We can use Overspill again and $(*)$ to obtain a nonstandard $H \in \mathbb{N}^{\mathfrak{M}}$ and some d in $\mathfrak{M}$ such that

$$\mathfrak{M} \models \forall i < H\ (\theta(i, d) \leftrightarrow i \in d).$$

This shows that for all $i \in \omega$, $\mathfrak{M} \models (\theta_i(d) \leftrightarrow i \in d)$. Therefore if $\mathfrak{M}$ is recursive, then so is the set of standard numbers in d, contradicting the recursive inseparability of A, B. $\square$

Before presenting the next result, let us observe that *the first order theory of a digraph G does not, in general, determine the first order theory of* $\mathbb{V}_\omega(G)$. To see this, let $2 \times G_\omega$ be the disjoint sum of two copies of the digraph G_ω of Example 3.6(b). Then G_ω and $2 \times G_\omega$ are elementarily equivalent, but $\mathbb{V}_\omega(G_\omega)$ and $\mathbb{V}_\omega(2 \times G_\omega)$ are not elementarily equivalent, indeed, both $\mathbb{V}_\omega(G_\omega)$ and $\mathbb{V}_\omega(2 \times G_\omega)$ are the unique ω-models of their own first order theories up to isomorphism. However, Theorem 3.12 shows that the $L_{\infty\omega}$-theory of $\mathbb{V}_\omega(G)$ is determined by the $L_{\infty\omega}$-theory of G.

Recall that the infinitary logic $L_{\infty\omega}$ is the extension of first logic that allows the formation of disjunctions and conjunctions of arbitrary cardinality, but which only uses finite strings of quantifiers. By a classical theorem of Karp [Ba, Ch. VII, Thm 5.3], two structures $\mathfrak{A}$ and $\mathfrak{B}$ are $L_{\infty\omega}$ equivalent iff $\mathfrak{A}$ and $\mathfrak{B}$ are *partially isomorphic*, i.e., there is a collection I of partial isomorphisms between $\mathfrak{A}$ and $\mathfrak{B}$ with the back-and-forth-property (written $I : \mathfrak{A} \cong_p \mathfrak{B}$). More specifically, $I : \mathfrak{A} \cong_p \mathfrak{B}$ if each $f \in I$ is an isomorphism between some substructure $\mathfrak{A}_f$ of $\mathfrak{A}$ and some substructure $\mathfrak{B}_f$ of $\mathfrak{B}$; and for every $f \in I$ and every $a \in A$ (or $b \in B$) there is a $g \in I$ with $f \subseteq g$ and $a \in \mathsf{dom}(g)$ (or $b \in \mathsf{ran}(g)$, respectively).

Theorem 3.12. *Suppose G and G' are digraphs with $G \equiv_{\infty\omega} G'$. Then $\mathbb{V}_\omega(G) \equiv_{\infty\omega} \mathbb{V}_\omega(G')$.*

Proof: By Karp's Theorem, it suffices to show that if $G \cong_p G'$, then $\mathbb{V}_\omega(G) \cong_p \mathbb{V}_\omega(G')$.[18] Let $I : G \cong_p G'$ and consider the sequence $\{I_n : n \in \omega\}$, with $I_0 := I$ and $I_{n+1} = \{\overline{f} : f \in I_n\}$, where

$$\overline{f} := f \cup \{\langle S, f(S)\rangle : S \in D(\mathbb{V}_n(G)) \cap \mathsf{dom}(f)\}.$$

We claim that for each $n \in \omega$, $I_n : \mathbb{V}_n(G) \cong_p \mathbb{V}_n(G')$. Observe that if $f \in I_n : \mathbb{V}_n(G) \cong_p \mathbb{V}_n(G)$, and $S \in D(V_n(G_1))$, then $f(S) \in D(\mathbb{V}_n(G'))$. Using this observation it is easy to show that each member of I_n is a partial isomorphism between a substructure of $\mathbb{V}_n(G)$ and a substructure of $\mathbb{V}_n(G')$.

The back-and-forth property of I_n is established by induction. Since the base case is true by our choice of I_0, and G and G' play a symmetric role in the proof, it suffices to verify the "forth" portion of the inductive clause by showing that the forth-property is preserved in the passage from I_n to I_{n+1}. So suppose that $\overline{f} \in I_{n+1}$ (where $f \in I_n$), and $a \in V_{n+1}(G)$. We need to find $g \in I_n$ such that $\overline{g}$ extends $\overline{f}$ and $a \in \mathsf{dom}(\overline{g})$. We distinguish two cases:

- Case 1: $a \in V_n(G)$.
- Case 2: $a \in V_{n+1}(G) \backslash V_n(G)$.

If Case 1 holds, then the desired g exists by the forth-property of I_n. On the other hand, if Case 2 holds, then a is one of the deficiency sets

[18] Advanced methods in set theory provide a succinct proof of this fact. If $G \cong_p G'$, then there is a Boolean extension $\mathbf{V}^{\mathbb{B}}$ of the universe of set theory wherein $G \cong G'$, which in turn implies that $\mathbb{V}_\omega[G] \cong \mathbb{V}_\omega[G']$ in $\mathbf{V}^{\mathbb{B}}$. But since $\mathbb{V}_\omega[G]$ is absolute in the passage to a Boolean extension for every G, and $L_{\infty\omega}$-equivalence is a Π_1-notion, $\mathbb{V}_\omega[G] \equiv_{\infty\omega} \mathbb{V}_\omega[G']$ in the real world.

of $\mathbb{V}_n(G)$, and

$$a = \{c_1, \cdots, c_k\} \subseteq V_n(G).$$

Therefore by k-applications of the "forth" property of I_n, we can find an extension g of f with $\{c_1, \cdots, c_k\} \subseteq \mathsf{dom}(g)$. This will ensure that $\overline{g}$ extends $\overline{f}$ and $a \in \mathsf{dom}(\overline{g})$. Having constructed the desired $\{I_n : n \in \omega\}$, we let $I_\omega := \bigcup_{n\in\omega} I_n$. Clearly

$$I_\omega : \mathbb{V}_\omega(G) \cong_p \mathbb{V}_\omega(G').$$

□

Example 3.13. Suppose I and J are infinite sets and G is a digraph. Then $I \times G \cong_p J \times G$. To see this, we can choose the corresponding I to consist of (full) isomorphisms between structures of the form $X \times G$ and $Y \times G$, where X and Y are finite subsets of I and J (respectively) of the same finite cardinality. In particular, for any digraph G

$$\mathbb{V}_\omega(I \times G) \equiv_{\infty\omega} \mathbb{V}_\omega(J \times G).$$

Remark 3.14.

(a) Let $\mathcal{G}$ be the category whose *objects* are extensional acyclic digraphs all of whose vertices have finite in-degree, and whose *morphisms* are end embeddings, i.e., embeddings $f : G \to G'$ with the property that $f(G) \subseteq_{\mathsf{e}} G'$; and let $\mathcal{G}_{\mathsf{ZF_{fin}}}$ be the subcategory of $\mathcal{G}$ whose objects are ω-models of $\mathsf{ZF_{fin}}$. Then there is a functor

$$\Phi : \mathcal{G} \to \mathcal{G}_{\mathsf{ZF_{fin}}}.$$

Moreover, Φ is a retraction (i.e., if $G \in \mathcal{G}_{\mathsf{ZF_{fin}}}$, then $\Phi(G) \cong G$), and the following diagram commutes (in the diagram η is the inclusion map)

$$\begin{array}{ccc} G & \xrightarrow{f} & G' \\ \downarrow \eta & & \downarrow \eta \\ \mathbb{V}_\omega(G) & \xrightarrow{\Phi(f)} & \mathbb{V}_\omega(G') \end{array}$$

Φ is defined in an obvious manner: $\Phi(G) := \mathbb{V}_\omega(G)$; and for each end embedding $f : G \to G'$, $\Phi(f)$ is recursively constructed by $\Phi(f)(x) = f(x)$ for all $x \in G$, and for $S \in V_{n+1}(G)\backslash V_n(G)$, $\Phi(f)(S) =$ the unique element v of $\mathbb{V}_\omega(G')$ such that $v_F = \{f(x) : x \in S\}$, where F is the membership relation of $\mathbb{V}_\omega(G')$. Note that $\Phi(f)$ is the unique extension of f to a morphism whose domain is $\mathbb{V}_\omega(G)$. Indeed, it is not hard to see that Φ is the *left adjoint* of the functor e that identically embeds $\mathcal{G}_{\mathsf{ZF_{fin}}}$ into $\mathcal{G}$. Since each functor has at most one left adjoint up to natural isomorphism [Mac, Cor. 1, Ch. IV], this shows that Φ is a closure operation that can be characterized in an "implementation-free" manner.

(b) It is easy to see that for each morphism $f : G \to G'$ of $\mathcal{G}$, $\Phi(f)$ is surjective if f is surjective. Therefore, $\Phi(f)$ is an automorphism of $\mathbb{V}_\omega(G)$ if f is an automorphism of G. Indeed, for any fixed $G \in \mathcal{G}$, the map $f \mapsto \Phi(f)$ defines a *group embedding* from $\mathsf{Aut}(G)$ into $\mathsf{Aut}(\mathbb{V}_\omega(G))$. Note that $\Phi(f)$ is the only automorphism of $\mathbb{V}_\omega(G)$ that extends f, since if g is any automorphism of $\mathbb{V}_\omega(G)$, then for each $S \in \mathbb{V}_\omega(G)\backslash G$, by $\mathsf{Extensionality}$, $g(S) = \{g(x) : x \in S\}$. In general, $\mathsf{Aut}(G)$ need not be isomorphic to $\mathsf{Aut}(\mathbb{V}_\omega(G))$, e.g., G_ω of Example 3.6(b) is a rigid digraph, but $\mathsf{Aut}(\mathbb{V}_\omega(G_\omega))$ is an infinite cyclic group. However, Theorem 4.2 shows that $\mathsf{Aut}(G) \cong \mathsf{Aut}(\mathbb{V}_\omega(G)$ for a wide class of digraphs G.

4. MODELS WITH SPECIAL PROPERTIES

In this section we refine the method introduced in the proof of Theorem 3.4 in order to construct various large families of ω-models of $\mathsf{ZF}_{\mathsf{fin}}$ with a variety of additional structural features, e.g., having a prescribed automorphism group, or being pointwise definable. The central result of this section is Theorem 4.2 below which shows that one can canonically code any prescribed graph (A, F), where F is a set of 2-element subsets of A, into a model of $\mathsf{ZF}_{\mathsf{fin}}$, thereby yielding a great deal of control over the resulting ω-models of $\mathsf{ZF}_{\mathsf{fin}}$.

Definition 4.1. In what follows, every digraph $G = (X, E)$ we consider will be such that $\omega \subseteq X$; and for $n \in \omega$, $n_E = \{0, 1, \ldots, n-1\}$. In particular, $0_E = \emptyset$.

(a) Let $\varphi_1(x)$ be the formula in the language of set theory that expresses

"there is a sequence $\langle x_n : n < \omega\rangle$ with $x = x_0$ such that $x_n = \{n, x_{n+1}\}$ for all $n \in \omega$".

At first sight, this seems to require an existential quantifier over infinite sequences (which is not possible in $\mathsf{ZF}_{\mathsf{fin}}$), but this problem can be circumvented by writing $\varphi_1(x)$ as the sentence that expresses the equivalent statement

"for every positive $n \in \omega$, there is a sequence $s_n = \langle x_i : i < n\rangle$ with $x_0 = x$ such that for all $i < n-1$, $x_i = \{i, x_{i+1}\}$".

(b) Using the above circumlocution, let $\theta(x, y, e)$ be the formula in the language of set theory that expresses

"$x \neq y \wedge \varphi_1(x) \wedge \varphi_1(y)$, with corresponding sequences $\langle x_n : n \in \omega \rangle$, and $\langle y_n : n \in \omega \rangle$, and there is a sequence $\langle e_n : n \in \omega \rangle$ with $e = e_0$ such that $e_n = \{e_{n+1}, x_n, y_n\}$ for all $n \in \omega$."

Then let $\varphi_2(x, y) := \exists e\, \theta(x, y, e)$.

Theorem 4.2. *For every graph (A, F) there is an ω-model $\mathfrak{M}$ of $\mathsf{ZF}_{\mathsf{fin}}$ whose universe contains A and which satisfies the following conditions:*

(a) *(A, F) is definable in $\mathfrak{M}$;*
(b) *Every element of $\mathfrak{M}$ is definable in $(\mathfrak{M}, x)_{x \in A}$;*
(c) *If (A, F) is pointwise definable, then so is $\mathfrak{M}$;*
(d) $\mathsf{Aut}(\mathfrak{M}) \cong \mathsf{Aut}(A, F)$.

Proof: Let

$$X = \omega \cup \big((A \cup F) \times \omega\big).$$

For elements of $X \backslash \omega$, we write z_n instead of $\langle z, n \rangle$. This notation should be suggestive of how these elements are used in the definitions of $\varphi_1(x, y)$ and $\theta(x, y, e)$. Let

$$\begin{aligned} E \;=\;& \{\langle m, n \rangle : m < n \in \omega\} \cup \\ & \{\langle n, x_n \rangle : n \in \omega,\ x \in A\} \cup \\ & \{\langle x_{n+1}, x_n \rangle : n \in \omega,\ x \in A\} \cup \\ & \{\langle e_{n+1}, e_n \rangle : n \in \omega,\ e \in F\} \cup \\ & \{\langle x_n, e_n \rangle : n \in \omega,\ x \in e \in F\}. \end{aligned}$$

Moreover, in order to arrange $A \subseteq X$ we can identify elements of the form $\langle x, 0 \rangle$ with x when $x \in A$. This gives $G = (X, E)$, which is an extensional, acyclic digraph all of whose vertices have finite in-degree. Let $\mathfrak{M} := \mathbb{V}_\omega(G)$. By Theorem 3.4, $\mathfrak{M}$ is an ω-model of $\mathsf{ZF}_{\mathsf{fin}}$. Note that if $n \in \omega$, $x \in A$, and $e = \{x, y\} \in F$, then $\mathfrak{M}$ satisfies both "$x_n = \{x_{n+1}, n\}$" and "$e_n = \{x_n, y_n, e_{n+1}\}$", as shown in Figure 1.

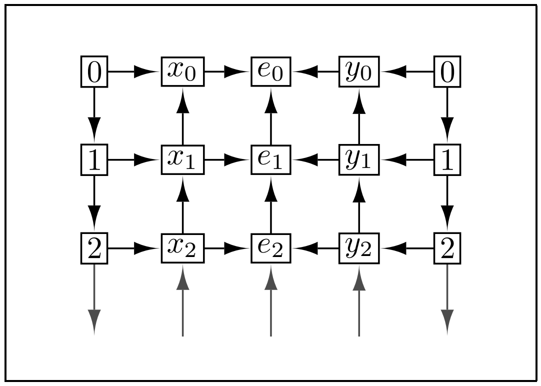

Figure 1: Representation of an Edge

It is clear that if $\{x, y\} \in F$, then $\mathfrak{M} \models \varphi_2(x, y)$. In order to establish (a) it suffices to check that if $\mathfrak{M} \models \varphi_1(x)$, then $x \in A$, i.e., we need to verify that $V_\omega(G) \setminus X$ does not contain an element that satisfies φ_1. Suppose $\mathfrak{M} \models \varphi_1(c)$ for some $c \in V_\omega(G) \setminus X$. Then there is a sequence $\langle c_n : n < \omega \rangle$ with $c = c_0$ such that $c_n = \{n, c_{n+1}\}$ for all $n \in \omega$. On the other hand, by Remark 3.5(b), we may choose $n_0 > 0$ as the first $n \in \omega$ for which $\tau(c) \backslash \tau(n, c) \subseteq G$. It is easy to see that this allows us to find some $x \in A$ such that $\tau(c) \backslash \tau(n, c) = \tau(x) \backslash \tau(n, x)$. Since $n_0 > 0$ this in turn shows that $\{n_0, c_{n_0+1}\} = \{n_0, x_{n_0+1}\}$ which implies that $\tau(c) \backslash \tau(n_0 - 1, c) \subseteq G$, thereby contradicting the minimality of n_0. This concludes the proof of (a).

In light of Remark 3.5(a), in order to establish (b) it suffices to show that every vertex of G is definable in $(\mathfrak{M}, x)_{x \in A}$. Since it is clear that each element of ω is definable in $\mathfrak{M}$, we shall focus on the definability of elements of $(A \cup F) \times \omega$. Each x_n is definable in $(\mathfrak{M}, x_0)$ since $\mathfrak{M}$ satisfies "$x_n = \{n, x_{n+1}\}$". Similarly, since $\mathfrak{M}$ satisfies "$e_n = \{e_{n+1}, x_n, y_n\}$" the definability of each e_n follows from the definability of e_0 in $(\mathfrak{M}, x)_{x \in A}$. To verify this, suppose $e = \{x, y\}$. Let θ be as in Definition 4.1. Then $\theta(x, y, z)$ defines e_0 in $(\mathfrak{M}, x, y)$.

(c) is an immediate consequence of (a) and (b) since if (A, F) is pointwise definable, then by (a), every element of A is definable in $\mathfrak{M}$.

To prove (d), we first establish $\mathsf{Aut}(G) \cong \mathsf{Aut}(\mathfrak{M})$, where G is the digraph (X, E). We already commented in Remark 3.14(b) that (1) there is an embedding Φ from $\mathsf{Aut}(G)$ into $\mathsf{Aut}(\mathfrak{M})$, and (2) every automorphism of G has a *unique* extension to an automorphism of $\mathfrak{M}$. Therefore, in order to show that Φ is surjective, it suffices to show that X is definable in $\mathfrak{M}$ since the definability of X in $\mathfrak{M}$ would imply that $g \restriction X \in \mathsf{Aut}(G)$ for every $g \in \mathsf{Aut}(\mathfrak{M})$. Recall that φ_1 defines $\{x_0 : x \in A\}$ in $\mathfrak{M}$. On the other hand,

$$\varphi_3(z) := \exists x \exists y \ \theta(x, y, z)$$

defines $\{e_0 : e \in F\}$ in $\mathfrak{M}$. So X is definable in $\mathfrak{M}$ by the formula

$$\varphi_4(u) := \mathbb{N}(u) \vee \exists v \left(u \in \tau(v) \wedge (\varphi_1(v) \vee \varphi_3(v))\right).$$

This concludes the proof of $\mathsf{Aut}(G) \cong \mathsf{Aut}(\mathfrak{M})$. Therefore to establish (c) it suffices to verify that $\mathsf{Aut}(A, F) \cong \mathsf{Aut}(G)$. Suppose $f \in \mathsf{Aut}(A, F)$. We shall build $\overline{f} \in \mathsf{Aut}(G)$ such that $f \mapsto \overline{f}$ describes an isomorphism between $\mathsf{Aut}(A, F)$ and $\mathsf{Aut}(G)$. $\overline{f}$ is defined by cases:

- $\overline{f}(n) = n$ for $n \in \omega$.
- $\overline{f}$ is defined recursively on $\{x_n : x \in A,\ n \in \omega\}$ by: $\overline{f}(x_0) = f(x_0)$ and $\overline{f}(x_{n+1})$ = the unique element $v \in G$ for which $(f(x_n))_E = \{n, v\}$.
- $\overline{f}$ is defined recursively on $\{e_n : e \in F,\ n \in \omega\}$ by: for $e = \{x, y\}$, $\overline{f}(e_0) = e'_0$, where $e' = \{f(x), f(y)\}$, and $\overline{f}(e_{n+1})$ = the unique element $v \in G$ for which $\left(\overline{f}(e_n)\right)_E = \{\overline{f}(x_n), \overline{f}(y_n), v\}$.

Moreover, it is easy to see that if $g \in \mathsf{Aut}(G)$, then g is uniquely determined by $g \restriction A$. Therefore in order to verify that the map $f \mapsto \overline{f}$ is surjective, it suffices to observe that A is definable in G by the formula $\psi(x)$ that expresses "$0 \in x \wedge |x| = 2 \wedge x \neq \{0, 1\}$". This shows that $\mathsf{Aut}(A, F) \cong \mathsf{Aut}(G)$ and completes the proof of (d).

□

Remark 4.3.

(a) Let $\mathcal{G}_{\mathsf{ZF_{fin}}}$ be the category defined in Remark 3.14, and consider the category $\mathcal{F}$ whose *objects* are graphs (A, F), and whose *morphisms* are embeddings $f : (A, F) \to (A', F')$. The proof of Theorem 4.2 shows that there is a functor Ψ from $\mathcal{F}$ into $\mathcal{G}_{\mathsf{ZF_{fin}}}$ such that for every object $\mathcal{A} = (A, F)$ of $\mathcal{F}$, the set of vertices of $\Psi(\mathcal{A})$ includes A; every automorphism of $\mathcal{A}$ has a unique extension to an automorphism of $\Psi(\mathcal{A})$; and every automorphism of an object in $\mathcal{G}_{\mathsf{ZF_{fin}}}$ of the form $\Psi(\mathcal{A})$ is uniquely determined by its restriction to A.

(b) Let φ_1 and θ be as in Definition 4.1, and let $\delta(z)$ be the formula in the language of set theory that expresses:

"z is of the form $\{X, E\}$, where $\varphi_1(x)$ holds for each $x \in X$, and for each $e \in E$, there are x and y in X such that $\theta(x, y, e)$",

and let $\sigma := \forall t \exists z(t \in \tau(z) \wedge \delta(z))$. Then the class of ω-models of $\mathsf{ZF_{fin}} + \sigma$ are precisely those objects in $\mathcal{F}$ that lie in the range of the functor Ψ defined in (a) above. Note that $(V_\omega, \in) \models \sigma$ since $\delta(\{\emptyset\})$ vacuously holds in $(V_\omega, \in)$.

Corollary 4.4. *Every group can be realized as the automorphism group of an ω-model of $\mathsf{ZF_{fin}}$.* [19]

Proof: A classical theorem of Frucht [Fr] shows that every group can be realized as the automorphism group of a graph. [20] $\square$

Corollary 4.5. *For every infinite cardinal κ there are 2^κ nonisomorphic rigid ω-models of $\mathsf{ZF_{fin}}$ of cardinality κ.*

Proof: It is well-known that there are 2^κ nonisomorphic rigid graphs of cardinality κ. One way to see this is to first show that there are 2^κ nonisomorphic rigid linear orders of cardinality κ by the following construction: start with the well-ordering $(\kappa, \in)$, and for each $S \subseteq \kappa$ let $\mathbb{L}_S$ be the linear order obtained by inserting a copy of ω^* (the order-type of negative integers) between α and $\alpha + 1$ for each $\alpha \in S$. Since $\mathbb{L}_S$ and $\mathbb{L}_{S'}$ are nonisomorphic for $S \neq S'$ this completes the argument since every linear order can be coded into a graph with the same automorphism group: given a linear order $(L, <_L)$, let $[L]^2$ be the set of all 2-element subsets of L, and for each $s \in [L]^2$, introduce distinct vertices $\{a_s, b_s\}$, and consider the graph (A, F), where

$$A = L \cup \{a_s : s \in S\} \cup \{b_s : s \in S\},$$

and

$$\begin{aligned} F \quad = \quad & \{\{x, a_s\} : x \in s \in [L]^2\} \cup \\ & \{\{a_s, b_s\} : s \in [L]^2\} \cup \\ & \{\{x, b_s\} : s = \{x, y\}, \text{ and } x <_L y\}. \end{aligned}$$

$\square$

Corollary 4.6. *For every infinite cardinal κ there is a family $\mathcal{M}$ of cardinality 2^κ* of *ω-models of $\mathsf{ZF_{fin}}$ of cardinality κ such that for any*

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[19] In contrast, the class of groups that arise as automorphism groups of models of $\mathsf{ZF_{fin}} + \mathsf{TC}$ are precisely the right-orderable groups. This is a consequence of coupling the bi-interpretability of $\mathsf{ZF_{fin}} + \mathsf{TC}$ and PA with a key result [KS, Theorem 5.4.4] about automorphisms of models of PA.

[20] See [Lo] for an exposition of Frucht's theorem and its generalizations.

distinct $\mathfrak{M}_1$ *and* $\mathfrak{M}_2$ *in* $\mathcal{M}$, *there is no elementary embedding from* $\mathfrak{M}_1$ *into* $\mathfrak{M}_2$.

Proof: It is well-known that there is a family $\mathcal{G}$ of power 2^κ of simple graphs of cardinality κ such that for distinct G_1 and G_2 in $\mathcal{G}$, there is no embedding from G_1 into G_2.[21] □

The next corollary is motivated by the following observations: (1) $(V_\omega, \in)$ is a pointwise definable model, and (2) The Gödel-Rosser incompleteness theorem is powerless in giving any information on the number of complete extensions of $\mathsf{ZF}_{\mathsf{fin}}$ that possess an ω-model.[22]

Corollary 4.7. *There are* $2^{\aleph_0}$ *pointwise definable* ω*-models of* $\mathsf{ZF}_{\mathsf{fin}}$. *Consequently there are* $2^{\aleph_0}$ *complete extensions of* $\mathsf{ZF}_{\mathsf{fin}}$ *that possess* ω*-models.*

Proof: This is an immediate consequence of Theorem 4.2(c) and the well-known fact that there are $2^{\aleph_0}$ nonisomorphic pointwise definable graphs. One way to establish the latter fact is as follows: for each $S \subseteq \omega$ first build a graph $\mathcal{A}_S$ with the property that no two vertices have the same degree, each vertex has finite degree, and if $n < \omega$, then $n \in S$ iff there is a vertex having degree n. Clearly $\mathcal{A}_S$ is pointwise definable, and distinct S's yield nonisomorphic $\mathcal{A}_S$ (incidentally, if $0 \notin S$, then $\mathcal{A}_S$ can be arranged to be connected. Also, if S is r.e., then a highly recursive $\mathcal{A}_S$ can be constructed). □

The next corollary is an immediate consequence of coupling Theorem 4.2 with a classical construction [Ho, Theorem 5.5.1] that implies that for every structure $\mathfrak{A}$ in a *finite* signature there is a graph $G_{\mathfrak{A}}$ that is bi-interpretable with an isomorphic copy of $\mathfrak{A}$. It is easy to see (but a bit cumbersome to write out the details of the proof) that if two structures $\mathfrak{A}$ and $\mathfrak{B}$ are bi-interpretable, then (1) their automorphism groups $\mathsf{Aut}(\mathfrak{A})$ and $\mathsf{Aut}(\mathfrak{B})$ are isomorphic [Ho, Exercise 8, Sec. 5.4], and (2) $\mathfrak{A}$ is pointwise definable iff $\mathfrak{B}$ is pointwise definable. We could have used this approach to provide succinct indirect proofs of the above

[21]Indeed, by a theorem of Nešetřil and Pultr [NP] one can stengthen this result by replacing "embedding" to "homomorphism". We should also point out that by a result of Perminov [P] for each infinite cardinal κ, there are 2^κ nonisomorphic rigid graphs none of which can be embedded in to any of the others. This shows that Corollaries 4.6 and 4.7 can be dovetailed.

[22]The Gödel-Rosser theorem can be fine-tuned to show the essential undeciability of consistent first order theories that interpret Robinson's Q [HP, Thm 2.10, Chap. III], thereby showing that such theories have continuum-many consistent completions (indeed, the result continues to hold with Q replaced by the weaker system R).

corollaries, but in the interest of perspicuity, we opted for more direct proofs.

Corollary 4.8. *For every structure $\mathfrak{A}$ in a finite signature there is an ω-model $\mathfrak{M}$ of $\mathsf{ZF_{fin}}$ such that $\mathfrak{M}$ interprets an isomorphic copy of $\mathfrak{A}$, and $\mathsf{Aut}(\mathfrak{A}) \cong \mathsf{Aut}(\mathfrak{B})$. Furthermore, if $\mathfrak{A}$ is pointwise definable, then so is $\mathfrak{M}$.*

To motivate the last result of this section, let us recall that V_ω is the only ω-model of $\mathsf{ZF_{fin}} + \mathsf{TC}$ up to isomorphism. As shown by Corollary 4.8 below, there are many other "categorical" *finite* extensions of $\mathsf{ZF_{fin}}$. One can also show that there are continuum-many completions of $\mathsf{ZF_{fin}}$ that possess a unique ω-model up to isomorphism.

Corollary 4.9. *There are infinitely many countable nonisomorphic ω-models of $\mathsf{ZF_{fin}}$ each of which is the unique ω-model of some finite extension of $\mathsf{ZF_{fin}}$.*

Proof: For $0 < r \in \omega$, let $\psi_r(x)$ be the following variant of $\varphi_1(x)$ of Definition 4.1:

"there is a sequence $\langle x_n : n < \omega\rangle$ with $x = x_0$ such that $x_n = \{n, x_{n+1}\}$ if $r \mid n$; and $x_n = \{x_{n+1}\}$ otherwise."

Next, let θ_r be the sentence that expresses

$$\text{“}\exists x\ (V = V_\omega((\tau(x), \in)) \text{ and } \psi_r(x)\text{”}.$$

Clearly $\mathsf{ZF_{fin}} + \theta_r$ has a unique ω-model up to isomorphism, and for $r \neq s$, no ω-model of $\mathsf{ZF_{fin}}$ satisfies both θ_r and θ_s. □

Remark 4.10. The theories $\mathsf{ZF_{fin}} + \theta_r$ of the proof of Corollary 4.9 provide examples of extensions of $\mathsf{ZF_{fin}} + \neg\mathsf{TC}$ that are bi-interpretable with PA. Furthermore, for every positive r, every model of PA has a unique extension to a model of $\mathsf{ZF_{fin}} + \theta_r$, i.e., if $\mathfrak{M}$ and $\mathfrak{N}$ are models of $\mathsf{ZF_{fin}} + \theta_r$ such that there is an isomorphism f between $(\mathbb{N},+,\times)^{\mathfrak{M}}$ and $(\mathbb{N},+,\times)^{\mathfrak{N}}$, then f has a (unique) extension to an isomorphism between $\mathfrak{M}$ and $\mathfrak{N}$.

5. $\mathsf{ZF_{fin}}$ AND PA ARE NOT BI-INTERPRETABLE

In this section we wish to carry out the promise made at the end of Remark 2.2(f) by establishing the strong failure of bi-interpretability of $\mathsf{ZF_{fin}}$ and PA. Recall that two theories U and V are said to be *bi-interpretable* if there are interpretations $\mathcal{I} : U \to V$ and $\mathcal{J} : V \to U$, a binary U-formula F, and a binary V-formula G, such that F is,

U-verifiably, an isomorphism between id_U and $\mathcal{J} \circ \mathcal{I}$, and G is, V-verifiably, an isomorphism between id_V and $\mathcal{I}\circ\mathcal{J}$. This notion is entirely syntactic, but has several model theoretic ramifications. In particular, given models $\mathfrak{A} \models U$ and $\mathfrak{B} \models V$, the interpretations $\mathcal{I}$ and $\mathcal{J}$ give rise to (1) models $\mathcal{J}^{\mathfrak{A}} \models V$ and $\mathcal{I}^{\mathfrak{B}} \models U$, and (2) isomorphisms $F^{\mathfrak{A}}$ and $G^{\mathfrak{B}}$ with

$$F^{\mathfrak{A}} : \mathfrak{A} \longrightarrow (\mathcal{J} \circ \mathcal{I})^{\mathfrak{A}} = \mathcal{I}^{\left(\mathcal{J}^{\mathfrak{A}}\right)} \text{ and } G^{\mathfrak{B}} : \mathfrak{B} \xrightarrow{\cong} (\mathcal{I} \circ \mathcal{J})^{\mathfrak{B}} = \mathcal{J}^{\left(\mathcal{I}^{\mathfrak{B}}\right)}.$$

A much weaker notion, dubbed *sentential equivalence*[23] in [Vi-2], is obtained by replacing the demand on the existence of definable isomorphisms with the requirement that the relevant models be elementarily equivalent, i.e., for any $\mathfrak{A} \models U$ and $\mathfrak{B} \models V$,

$$\mathfrak{A} \equiv (\mathcal{J} \circ \mathcal{I})^{\mathfrak{A}} \text{ and } \mathfrak{B} \equiv (\mathcal{I} \circ \mathcal{J})^{\mathfrak{B}}.$$

Theorem 5.1. $\mathsf{ZF}_{\mathsf{fin}}$ *and* PA *are not sententially equivalent.*

Proof: Suppose to the contrary that the interpretations

$$\mathcal{I} : \mathsf{ZF}_{\mathsf{fin}} \to \mathsf{PA}, \text{ and } \mathcal{J} : \mathsf{PA} \to \mathsf{ZF}_{\mathsf{fin}}$$

witness the sentential equivalence of $\mathsf{ZF}_{\mathsf{fin}}$ and PA. In light of the fact that there are at least two elementarily inequivalent recursive models of $\mathsf{ZF}_{\mathsf{fin}}$, in order to reach a contradiction it is sufficient to demonstrate that the hypothesis about $\mathcal{I}$ and $\mathcal{J}$ can be used to show that any two arithmetical ω-models of $\mathsf{ZF}_{\mathsf{fin}}$ are elementarily equivalent. To this end, suppose $\mathfrak{M}$ is an arithmetical ω-model of $\mathsf{ZF}_{\mathsf{fin}}$. We claim that

$$\mathcal{J}^{\mathfrak{M}} \cong (\omega, +, \times).$$

The following classical theorem of Scott[24] plays a key role in the verification of our claim. In what follows, an *arithmetical model of* PA refers to a structure of the form $(\omega, \oplus, \otimes)$, where there are first order formulas $\varphi(x, y, z)$ and $\psi(x, y, z)$ that respectively define the graphs of the binary operations $\oplus$ and $\otimes$ in the model $(\omega, +, \times)$.

> Theorem (Scott [Sco]). *No arithmetical nonstandard model of* PA *is elementarily equivalent to* $(\omega, +, \times)$.

[23]It is easy to see (using the completeness theorem for first order logic) that sentential equivalence can also be *syntactically* formulated.

[24]In order to make the proof self-contained, we sketch an outline of the proof of Scott's theorem. If a nonstandard model $\mathfrak{A}$ is elementarily equivalent to $(\omega, +, \cdot)$, then the *standard system* $\mathsf{SSy}(\mathfrak{A})$ of $\mathfrak{A}$ has to include all arithmetical sets. If, in addition, $\mathfrak{A}$ were to be arithmetical, then every member of $\mathsf{SSy}(\mathfrak{A})$, and therefore every arithmetical set, would have to be of bounded quantifier complexity, contradiction. Scott's result has recently been revisited in [IT, Sec.3].

Let $\mathfrak{M}' \equiv (\mathcal{J} \circ \mathcal{I})^{\mathfrak{M}}$. By assumption, $\mathfrak{M}' \equiv \mathfrak{M}$, which implies that

$$(\mathbb{N},+,\times)^{\mathfrak{M}} \equiv (\mathbb{N},+,\times)^{\mathfrak{M}'}.$$

This shows that $(\mathbb{N},+,\times)^{\mathfrak{M}'}$ is elementarily equivalent to $(\omega,+,\times)$ since $\mathfrak{M}$ is an ω-model. Coupled with the fact that $(\mathbb{N},+,\times)^{\mathfrak{M}'}$ is an arithmetical model (since it is arithmetically interpretable in an arithmetical model), Scott's aforementioned theorem can be invoked to show that $\mathfrak{M}'$ must be an ω-model. But since no nonstandard model of PA can interpret an isomorphic copy of $(\omega,<)$ this shows that for any arithmetical ω-model $\mathfrak{M}$ of $\mathsf{ZF}_{\mathsf{fin}}$, $\mathcal{J}^{\mathfrak{M}} \cong (\omega,+,\times)$. Therefore, the assumption of sentential equivalence of $\mathsf{ZF}_{\mathsf{fin}}$ and PA implies that any two arithmetical ω-models of $\mathsf{ZF}_{\mathsf{fin}}$ are elementarily equivalent, which is the contradiction we were aiming to arrive at. $\square$

Remark 5.2.

(a) The proof of Theorem 5.1 only invoked one of the two elementary equivalences stipulated by the definition of sentential equivalence, namely that $\mathfrak{M} \equiv (\mathcal{J} \circ \mathcal{I})^{\mathfrak{M}}$ for every $\mathfrak{M} \models \mathsf{ZF}_{\mathsf{fin}}$. In the terminology of [Vi-2], this shows that $\mathsf{ZF}_{\mathsf{fin}}$ is not a retract of PA in the appropriate category in which definitional equivalence is isomorphism.

(b) As mentioned in Remark 3.10(b), there are infinitely many elementarily inequivalent recursive ω-models of $\mathsf{ZF}_{\mathsf{fin}} + \neg\mathsf{TC}$. Using the proof of Theorem 5.1 this fact can be used to show that the theories $\mathsf{ZF}_{\mathsf{fin}} + \neg\mathsf{TC}$ and PA are not sententially equivalent either. However, as pointed out in Remark 4.10, there are finite extensions of $\mathsf{ZF}_{\mathsf{fin}} + \neg\mathsf{TC}$ that are bi-interpretable with PA.

6. CONCLUDING REMARKS AND OPEN QUESTIONS

Let $\mathfrak{A} \equiv_n^{\mathsf{Levy}} \mathfrak{B}$ abbreviate the assertion that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same Σ_n^{Levy} formulas in the usual Levy hierarchy of formulas of set theory (in which only unbounded quantification is significant).

Theorem 6.1. *If $\mathfrak{A}$ and $\mathfrak{B}$ are ω-models of $\mathsf{ZF}_{\mathsf{fin}}$, then $\mathfrak{A} \equiv_1^{\mathsf{Levy}} \mathfrak{B}$.*

Proof: We use a variant of the Ehrenfeucht-Fraïssé game adapted to this context, in which once the first player (spoiler) and second player (duplicator) have made their first moves by choosing elements from each of the two structures, the players are obliged to select only members of elements that have been chosen already (by either party). We wish to show that for any particular length n of the play ($n > 0$), the duplicator has a winning strategy. Assume, without loss of generality, that the spoiler chooses $a_1 \in A$. The duplicator responds by choosing $b_1 \in B$ with the key property that there is function f such that

$$f : (\tau(n,a), \in)^{\mathfrak{A}} \xrightarrow{\cong} (\tau(n,b), \in)^{\mathfrak{B}}.$$

(it is easy to see, using the fact that $\mathfrak{A}$ and $\mathfrak{B}$ are ω-models of $\mathsf{ZF}_{\mathsf{fin}}$, that the duplicator can always do this). From that point on, once the spoiler picks any element c from either structure the duplicator responds with $f(c)$ or $f^{-1}(c)$ depending on whether c is in the domain or co-domain of f. □

Note that TC is Π_2^{Levy} (since $\mathsf{TC}(x)$ is Σ_1^{Levy}) relative to $\mathsf{ZF}_{\mathsf{fin}}$, so in the conclusion of Theorem 6.1, $\equiv_1^{\mathsf{Levy}}$ cannot be replaced by $\equiv_2^{\mathsf{Levy}}$ since $\mathfrak{A}$ can be chosen to be $(V_\omega, \in)$ and $\mathfrak{B}$ can be chosen to be a nonstandard ω-model of $\mathsf{ZF}_{\mathsf{fin}}$. These considerations motivate the next question.

Question 6.2. *Is it true that for each $n \geq 1$ there are ω-models $\mathfrak{A}$ and $\mathfrak{B}$ of $\mathsf{ZF}_{\mathsf{fin}}$ that are $\equiv_n^{\mathsf{Levy}}$ equivalent but not $\equiv_{n+1}^{\mathsf{Levy}}$ equivalent?*

Question 6.3. *For infinite cardinals κ and λ, let $\mathcal{M}(\kappa,\lambda)$ be the class of models $\mathfrak{M}$ of $\mathsf{ZF}_{\mathsf{fin}}$ such that the cardinality of $\mathfrak{M}$ is κ, and the cardinality of $\mathbb{N}^{\mathfrak{M}}$ is λ.*

(a) *Is there a first order scheme Γ_1 in the language of set theory such that $\mathsf{Th}(\mathcal{M}(\omega_1,\omega)) = \mathsf{ZF}_{\mathsf{fin}} + \Gamma_1$?*

(b) *Is there a first order scheme Γ_2 in the language of set theory such that $\mathsf{Th}(\mathcal{M}(\beth_\omega,\omega)) = \mathsf{ZF}_{\mathsf{fin}} + \Gamma_2$?*

Question 6.3 is motivated by two classical two-cardinal theorems of Model Theory[25] due to Vaught, which show that the answers to both parts of Question 6.3 are positive if "first order scheme" is weakened to "recursively enumerable set of first order sentences". The aforementioned two-cardinal theorems also show that (1) for $\kappa > \lambda \geq \omega$, $\mathsf{Th}(\mathcal{M}(\kappa,\lambda)) \supseteq \mathsf{Th}(\mathcal{M}(\omega_1,\omega))$, and (2) for all $\kappa \geq \beth_\omega$, $\mathsf{Th}(\mathcal{M}(\kappa,\omega)) \subseteq \mathsf{Th}(\mathcal{M}(\beth_\omega,\omega))$.

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[25]See [CK, Theorems 3.2.12. and 7.2.2], and [Sch-1].

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