AN INEVITABLE EXTENSION OF ZFC

Ali Enayat

Kunen Fest, April 3, 2009

Mahlo Cardinals

- A *Mahlo* cardinal κ is a strongly inaccessible cardinal such that the regular cardinals below κ form a *stationary* subset of κ.

- For an ordinal α, the α-*Mahlo cardinals* are defined recursively as follows:

  κ is 0-Mahlo if κ is strongly inaccessible;

  For $\alpha = \delta + 1$, κ is a α-Mahlo if

  $\{\gamma < \kappa : \gamma$ is δ-Mahlo$\}$ is stationary in κ;

  For limit α, κ is α-Mahlo if κ is δ-Mahlo for all $\delta < \alpha$.

Levy and Reflection

- Levy showed that Σ_n-truth is Σ_n-definable for $n \geq 1$ within ZF.

- In particular, for each natural number n there is a unary formula with the free variable α, denoted "$V_\alpha \prec_n \mathbf{V}$", that expresses:

for all Σ_n-formulae $\varphi(v_1, \cdots, v_k)$, and all $a_1, \cdots, a_k$ in V_α,

$$\varphi(a_1, \cdots, a_k) \leftrightarrow \varphi^{V_\alpha}(a_1, \cdots, a_k).$$

- For a unary formula $C(\alpha)$, possibly with suppressed parameters,

"$\{\alpha : C(\alpha)\}$ is c.u.b."

stands for the formula expressing

"$\{\alpha \in \mathbf{Ord} : C(\alpha)\}$ is c.u.b in $\mathbf{Ord}$".

Levy and Reflection, cont'd

- **Reflection Theorem** (Montague 1957; Levy 1960) *For each natural number* n, ZF *proves that* $\{\alpha : V_\alpha \prec_n \mathbf{V}\}$ *is c.u.b*.

- **Theorem** (Levy 1960). *For each natural numbers* n, *the following statement is provable within* ZF:

  (κ is (n+1)-Mahlo) $\rightarrow \exists\alpha < \kappa$ (α is n-Mahlo and $V_\alpha \prec V_\kappa$).

The Levy Scheme Λ

- $\lambda_{m,n}(\kappa)$ is the sentence in set theory asserting that κ is an m-Mahlo cardinal and
$$V_\kappa \prec_n \mathbf{V}.$$

- $\Lambda := \{\exists\kappa\ (\lambda_{n,n}(\kappa) : n \in \omega\}$.

- $\Lambda_1 := \{\forall\alpha \in \mathbf{Ord}\ \exists\kappa > \alpha\ \lambda_{n,n}(\kappa) : n \in \omega\}$.

- $\Lambda_2 := \{\forall\alpha \in \mathbf{Ord}\ \exists\kappa > \alpha\ \lambda_{m,n}(\kappa) : m \in \omega, n \in \omega\}$.

- $\Lambda_3 := \{\psi_{C(\alpha,x),n} : C = C(\alpha, x)$ is a binary formula of set theory$\}$, where
$$\psi_{C,n} := \forall x[\{\alpha \in \mathbf{Ord} : C(\alpha, x)\} \text{ is c.u.b.} \rightarrow \exists\kappa\ C(\kappa, x) \text{ and } \kappa \text{ is } n\text{-Mahlo}].$$

Different Faces of Λ

- **Theorem** (Levy 1960). *Over* ZF, *the theories* Λ, Λ_1, Λ_2, *and* Λ_3 *are pairwise equivalent*.

- $\Lambda_0 := \{\exists\kappa\ \kappa \text{ is } n\text{-Mahlo}: n \in \omega\}$.

- **Proposition** (Folklore)

  (a) *The theories* $\mathsf{ZF} + \Lambda_0$ *and* $\mathsf{ZF} + \Lambda$ *are equiconsistent.*

  (b) *Moreover, assuming Con(*$\mathsf{ZF}+\Lambda_0$*), neither* Λ_0*, nor* Λ *is finitely axiomatizable over* ZF.

The robustness of Λ

- **Theorem** *If* $M \models$ ZFC$+\Lambda$, *and* $c \in M$, *then* $(\mathbf{L(c)})^M \models \Lambda$.

- **Theorem** *If* $M \models$ ZFC $+ \Lambda$ *and* $\mathbb{P} \in M$ *is a partial order, then for every* $\mathbb{P}$*-generic filter* G *over* M, $M[G] \models \Lambda$.

- **Corollary.** *Suppose* Con*(*ZF$+\Lambda$*). Then for any sentence* ψ, Con(ZF+Λ+ψ) *if at least one of the following conditions are true:*

  (a) ZF $\vdash$ "*ψ holds in* $\mathbf{L}$", *or*

  (b) ZF $\vdash$ "*for some poset* $\mathbb{P}$, $1_{\mathbb{P}} \Vdash \psi$",

Finite Set Theory

- TC := "every set has a transitive closure".

- $\mathsf{ZF_{fin}} = \mathsf{ZF} \setminus \{\mathsf{Infinity}\} + \neg\mathsf{Infinity} + \mathsf{TC}$.

- $\mathsf{GBC_{fin}} = \mathsf{GBC} \setminus \{\mathsf{Infinity}\} + \neg\mathsf{Infinity} + \mathsf{TC}$.

- **Theorem** [Ackernann 1940, Kaye-Wong 2008]

  (a) $\mathsf{ZF_{fin}}$ *is bi-interpretable with* PA.

  (b) $\mathsf{GBC_{fin}}$ *is bi-interpretable with* $\mathsf{ACA_0}$.

Inevitability of Λ, Exhibit 1

- Let "Ord is WC" be the statement in class theory asserting that every "**Ord**-tree" has a branch of length **Ord**.

- **Theorem** [E 2004]

  (a) *If* $(M, \mathcal{A}) \models \mathsf{GBC} + \mathsf{Ord}$ is WC, *then* $M \models \mathsf{ZFC} + \Lambda$.

  (b) *Every completion of* $ZFC + \Lambda$ *has a countable model that has an expansion to a model of* $\mathsf{GBC} + \mathsf{Ord}$ is WC.

- **Corollary** $\mathsf{GBC}+\mathsf{Ord}$ is WC *is a conservative extension of* $\mathsf{ZFC} + \Lambda$.

- **Theorem** (Folklore) $\mathsf{GBC}_{\mathsf{fin}}$ *is a conservative extension of* $\mathsf{ZF}_{\mathsf{fin}}$.

Inevitability of Λ, Exhibit 2

- ZFC(I) is a theory in the language $\{\in, \mathbf{I}(x)\}$, where $\mathbf{I}(x)$ is a unary predicate.

- The axioms of ZFC(I) are as follows.

  (1) ZFC + All instances of replacement (hence separation) in $\{\in, \mathbf{I}(x)\}$;

  (2) $\mathbf{I}$ is a cofinal subclass of ordinals;

  (3) $\mathbf{I}$ is a class of indiscernibles for $(\mathbf{V}, \in)$.

Exhibit 2, Continued

- **Theorem** (E 2005). *The following are equivalent for a completion T of* ZFC*:*

  (1) T *has a model* M *that expands to a model* $(M, \mathbf{I}) \models$ ZFC(I).

  (2) T *has a model* M *that expands to* $(M, \mathbf{I}_n)_{n<\omega}$ *satisfying* ZF($\{\mathbf{I}_n : n \in \omega\}$) + "$\mathbf{I}_{n+1}$ *is a set of indiscernibles for* $(\mathbf{V}, \mathbf{I}_k)_{k \leq n}$".

  (3) T *is an extension of* ZFC + Λ.

- **Remark.** If Replacement (I) is weakened to Separation(I),the resulting system is conservative over ZFC.

- **Theorem** [E 2005] $\mathsf{ZF}_{\mathsf{fin}}$(I) *is a conservative extension of* $\mathsf{ZF}_{\mathsf{fin}}$.

Inevitability of Λ, Exhibit 3

- **Theorem**. [E 2001] *The Continuum Hypothesis is a sufficient, but not a necessary condition for every consistent extension of* ZF *to have an* $\aleph_2$*-like model.*

- **Theorem** [Kaufmann, E 1984] *Every completion of* ZFC *has a* θ*-like model for every* $\theta \geq \aleph_1$.

- **Theorem**. [E 2001] *Con(ZF +* there is an ω-Mahlo cardinal) *implies consistency of "the only completions of* ZFC *that have an* $\aleph_2$*-like model are those containing* Λ".

- **Theorem** (McDowell-Specker 1961). *Every completion of* $\mathsf{ZF}_{\mathsf{fin}}$ *has a* θ*-like model for every* $\theta \geq \aleph_1$.

Inevitability of Λ, Exhibit 4

- The theory NFU was introduced by Jensen as a modification of Quine's elegant formulation NF (New Foundations) of Russell's theory of types.

- NF is a first order theory whose axioms consist of the *stratifiable* comprehension scheme and the usual extensionality axiom.

- The stratifiable comprehension scheme is the collection of sentences of the form "$\{x : \varphi(x)\}$ exists", provided there is an integer valued function f whose domain is the set of all variables occurring in φ, which satisfies the following two requirements: (1) $f(v) + 1 = f(w)$, whenever $(v \in w)$ is a subformula of φ; (2) $f(v) = f(w)$, whenever $(v = w)$ is a subformula of φ.

Exhibit 4, Continued

- Jensen's variant NFU of NF is obtained by modifying the extensionality axiom so as to allow *urelements.*

- **Theorem** (Jensen 1968)

  (a) $\mathsf{Con(PA)} \Rightarrow \mathsf{Con(NFU} + \neg\mathsf{Infinity)}$.

  (b) $\mathsf{Con(Z)} \Rightarrow \mathsf{Con(NFU} + \mathsf{Choice} + \mathsf{Infinity)}$

- X is *Cantorian* if there is a one-to-one correspondence between X and $\{\{v\} : v \in X\}$; X is *strongly Cantorian* if the map sending v to $\{v\}$ (as v varies in X) exists;

- H := "every Cantorian set is strongly Cantorian"

- $\mathsf{NFUA} := \mathsf{NFU} + \mathsf{Infinity} + \mathsf{Choice} + \mathsf{H}$.

- $\mathsf{NFUA_{fin}} := \mathsf{NFUA} \backslash \{\mathsf{Infinity}\} + \{\neg\mathsf{Infinity}\}$.

Exhibit 4, Continued

- **Theorem** (Solovay, 1995) $\mathrm{Con}(\mathrm{ZFC}+\Lambda_0) \Leftrightarrow \mathrm{Con}(\mathrm{NFUA})$.

- **Theorem** (E 2002) *The following are equivalent for a theory T in the language $\{\in\}$:*

  (a) *T is a consistent completion of $ZFC+\Lambda$.*

  (b) *There is a model M* of *$NFUA$ such that $T = \mathrm{Th}(\text{"Cantorian part of V"})^M$.*

- **Theorem** (Solovay-E, 2002). *The analogue of the above theorem holds for* $\mathrm{ZF_{fin}}$ *and* $\mathrm{NFUA_{fin}}$, *in particular:*

$$\mathrm{Con}(\mathrm{NFUA_{fin}}) \Leftrightarrow \mathrm{Con}(\mathrm{ZF_{fin}}).$$

Inevitability of Λ, Exhibit 5

- EST is ZFC $\setminus\{\text{Power Set}, \text{Replacement}\}+\Delta_0$-Separation.

- GW is the axiom in the language $\{\in, \lhd\}$ that is the conjunction of the following 4 axioms:

  (1) $\lhd$ totally orders the universe; (2) Every nonempty set has a $\lhd$-least element, (3) $x \in y \rightarrow x \lhd y$; (4) $\forall x \exists y \forall z (z \in y \longleftrightarrow z \lhd x)$.

Exhibit 5, Continued

- **Theorem** [E 2004]

  (a) *For every completion* T *of* $\mathsf{ZFC} + \Lambda$ *there is a model* M_0 *of* $T + \mathsf{ZF}(\lhd) + \mathsf{GW}$ *such that* M_0 *has a proper e.e.e.* M *such that for some automorphism f of* M, *the fixed point set of f is* M_0.

  (b) *Moreover, if* j *is an automorphism of* $M \models \mathsf{EST}$ *whose fixed point set* M_0 *is a* $\lhd$-*initial segment of* N, *then* $M_0 \vDash \mathsf{ZFC} + \Lambda$.

- **Theorem**

  (a) (Gaifman) *The analogue of (a) above for* $\mathsf{ZF}_{\mathsf{fin}}$.

  (b) (E 2004) *The analogue of (b) above for* $\mathsf{ZF}_{\mathsf{fin}}$ *(with* $\mathsf{I}{-}\Delta_0$ *instead of* EST).