# Leibnizian Models of Set Theory*

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**Abstract**

A model is said to be *Leibnizian* if it has no pair of indiscernibles. Mycielski has shown that there is a first order axiom $LM$ (the Leibniz-Mycielski axiom) such that for any completion $T$ of Zermelo-Fraenkel set theory $ZF$, $T$ has a Leibnizian model iff $T$ proves $LM$. Here we prove:

**Theorem A.** *Every complete theory $T$ extending $ZF + LM$ has $2^{\aleph_0}$ nonisomorphic countable Leibnizian models.*

**Theorem B.** *If $\kappa$ is a prescribed definable infinite cardinal of a complete theory $T$ extending $ZF+$ $\mathbf{V} = \mathbf{OD}$, then there are $2^{\aleph_1}$ nonisomorphic Leibnizian models $\mathfrak{M}$ of $T$ of power $\aleph_1$ such that $(\kappa^+)^{\mathfrak{M}}$ is $\aleph_1$-like.*

**Theorem C.** *Every complete theory $T$ extending $ZF+$ $\mathbf{V} = \mathbf{OD}$ has $2^{\aleph_1}$ nonisomorphic $\aleph_1$-like Leibnizian models.*

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# 1. INTRODUCTION

Leibniz's well-known principle of the *identity of indiscernibles* [L, p. 308] suggests the following model theoretic definition: a structure $\mathfrak{M}$ in a first order language $\mathcal{L}$ is *Leibnizian* if $\mathfrak{M}$ contains no pair of indiscernibles, i.e., there are no distinct elements $a$ and $b$ in $\mathfrak{M}$ such that for every formula $\varphi(x)$ of $\mathcal{L}$ with one free variable $x$,

$$\mathfrak{M} \models \varphi(a) \leftrightarrow \varphi(b).$$

For example, the field $\mathbb{R}$ of real numbers is Leibnizian (since distinct real numbers have distinct Dedekind cuts[1]), but the field $\mathbb{C}$ of complex numbers is not (since $i$ and $-i$ are indiscernible). In this paper we build large families of countable and uncountable Leibnizian models of Zermelo-Fraenkel set theory $ZF$. Our source of inspiration was Mycielski's reflections on the foundations of set theory [My-1, My-2]. In particular, we were struck by Mycielski's discovery of a first order axiom[2] $LM$ in the language of set theory $\{\in\}$, which captures the spirit of Leibniz's dictum, as witnessed by the following result:

**Theorem 1** (Mycielski [My-1]). *A complete extension $T$ of $ZF$ proves $LM$ iff $T$ has a Leibnizian model.*

In Section 2 we refine Theorem 1 by showing that every completion $T$ of $ZF+LM$ has *continuum-many* nonisomorphic countable Leibnizian models. Surprisingly, the proof relies on two distinct model theoretic constructions, depending on whether or not the axiom $\mathbf{V} = \mathbf{OD}$ is proved by the theory $T$. In Section 3 we turn to the construction of *uncountable* Leibnizian models of set theory (note that a Leibnizian model in a countable language must have cardinality *at most* $2^{\aleph_0}$). More specifically, given a completion $T$ of $ZF + \mathbf{V} = \mathbf{OD}$, in Section 3.1 we build $2^{\aleph_1}$ nonisomorphic Leibnizian models $\mathfrak{M}$ of $T$ such that $(\kappa^+)^{\mathfrak{M}}$ is $\aleph_1$-like, where $\kappa$ is a prescribed $T$-definable infinite cardinal, while in Section 3.2 we use a very different method to build $2^{\aleph_1}$ nonisomorphic Leibnizian models of $T$ whose class of ordinals is $\aleph_1$-like. As pointed out in Remark 3.3.2(c), our work in Section 3.2 allows us to answer a question of Abramson and Harrington [AH]. We

[1] Indeed the same reasoning shows that every Archimedean ordered field is Leibnizian. Moreover, Tarski's elimination of quantifiers theorem for real closed fields implies that the *Leibnizian real closed fields are precisely the Archimedean real closed fields.* However, non-Archimedean Leibnizian ordered fields exist in every infinite cardinality $\leq 2^{\aleph_0}$.

[2] Mycielski called this axiom $A_2'$ in [My-1], and $E$ in [My-2]. The Leibniz-Mycielski appellation is proposed in [En-5], which probes the relationship between $LM$ and other axioms of set theory.

would have liked to construct uncountable Leibnizian models of all extensions of $ZF + LM$, but our techniques seem to work only for extensions of the stronger theory $ZF + \mathbf{V} = \mathbf{OD}$. Finally, in Section 4 we probe *well-founded* Leibnizian models of set theory.

I am grateful to Jan Mycielski for pointing the way, and to Robert Solovay for providing the key Lemma 3.1.3.

# PRELIMINARIES

Here we review a list of definitions and results about models of set theory which are central to this paper. Suppose $\mathfrak{M} = (M, E)$ is a model of $ZF$, where $E = \in^{\mathfrak{M}}$.

- $\mathbf{Ord}^{\mathfrak{M}}$ denotes the ordered set of "ordinals" of $\mathfrak{M}$ (ordered by $E$).
- For $c \in M$, $c_E = \{x \in M : xEc\}$.
- If $\mathfrak{M} \subseteq \mathfrak{N} = (N, F)$ and $c \in M$, then $\mathfrak{N}$ *fixes* $c$ if $c_E = c_F$, and $\mathfrak{N}$ *enlarges* $c$, if $c_E \subsetneq c_F$.
- $\mathfrak{N}$ *end extends* $\mathfrak{M}$, written $\mathfrak{M} \subseteq_e \mathfrak{N}$, if $\mathfrak{N}$ *fixes* every element of $\mathfrak{M}$.
- $\mathfrak{M}$ is an e.e.e. (elementary end extension) of $\mathfrak{M}$ if $\mathfrak{M} \prec \mathfrak{N}$ and $\mathfrak{N}$ end extends $\mathfrak{M}$. It is easy to see that if $\mathfrak{M} \prec_e \mathfrak{N} \models ZF$ then $\mathfrak{N}$ is a *rank extension* of $\mathfrak{M}$, i.e., the ordinal rank of every member of $N \backslash M$ (as computed in $\mathfrak{N}$) exceeds the ordinal rank of every member of $\mathfrak{M}$.
- Suppose $\kappa$ is a regular cardinal of $\mathfrak{M}$. $\mathfrak{N}$ is said to be a $\kappa$*-e.e.e.* if $\mathfrak{N}$ enlarges $\kappa$ but $\mathfrak{N}$ fixes every $\gamma \in \kappa_E$.
- The sentence $\mathbf{V} = \mathbf{OD}$ expresses - for models of $ZF$ - the statement "every set is first order definable from ordinal parameters". There is a parameter-free formula $\varphi(x, y)$ such that $ZF + \mathbf{V} = \mathbf{OD}$ proves "$\varphi$ well-orders the universe". Consequently, the theory $ZF + \mathbf{V} = \mathbf{OD}$ has definable Skolem functions, and every completion $T$ of $ZF + \mathbf{V} = \mathbf{OD}$ has a unique (up to isomorphism) model $\mathfrak{M}_T$ which is *pointwise definable* (i.e., every element of $\mathfrak{M}_T$ is definable in $\mathfrak{M}_T$ via a parameter-free formula).

- $\mathfrak{N}$ is a *minimal* elementary extension of $\mathfrak{M}$ iff $\mathfrak{M} \prec \mathfrak{N}$ and there is no model $\mathfrak{N}^*$ such that $\mathfrak{M} \prec \mathfrak{N}^* \prec \mathfrak{N}$. It is easy to see that for models of $ZF+\mathbf{V}=\mathbf{OD}$, $\mathfrak{N}$ is a minimal elementary extension of $\mathfrak{M}$, if $\mathfrak{M} \prec \mathfrak{N}$ and for any two elements $a$ and $b$ of $\mathfrak{N}$, there exists a definable term $\tau(x)$, possibly with parameters from $\mathfrak{M}$, such that $\mathfrak{N} \vDash \tau(a) = b$. This characterization relies on the availability of definable Skolem functions in models of $\mathbf{V}=\mathbf{OD}$.

- $\mathfrak{N}$ is a *minimal e.e.e.* of $\mathfrak{M}$ iff $\mathfrak{N}$ is a minimal elementary extension of $\mathfrak{M}$ and $\mathfrak{M} \prec_e \mathfrak{N}$. Similarly, $\mathfrak{N}$ is a *minimal $\kappa$-e.e.e.* of $\mathfrak{M}$ if $\mathfrak{N}$ is a minimal elementary extension of $\mathfrak{M}$ and $\mathfrak{N}$ is a $\kappa$-e.e.e. of $\mathfrak{M}$.

- The *Leibniz-Mycielski axiom LM*, is the following statement:

$$LM: \quad \forall x \forall y\ ([\forall \alpha > \max\{\rho(x), \rho(y)\}\ Th(V_\alpha, \in, x) = Th(V_\alpha, \in, y)] \rightarrow x = y).$$

Here $\rho(x)$ denotes the *ordinal rank* of $x$, $V_\alpha$ refers to the $\alpha$-th level of the von Neumann hierarchy consisting of sets of ordinal rank less than $\alpha$, and for structures $\mathfrak{A}$ of the form $(V_\alpha, \in, a)$, with $a \in V_\alpha$, $Th(\mathfrak{A})$ denotes the set of *sentences* in the language $\{\in, c\}$ that are true in $\mathfrak{A}$, where $c^{\mathfrak{A}} = a$. It is shown in [En-5] that in the presence of $ZF$, *$LM$ is equivalent to the existence of a parameter-free definable map* $\mathbf{G}$ *which injects the universe into the class of subsets of ordinals.* Consequently:

1. $ZF + \mathbf{V} = \mathbf{OD} \vdash LM$;

2. $ZF+$ $LM$ proves that the universe has a global linear ordering. In particular $ZF+$ $LM$ proves $AC_{<\omega}$ (the axiom of choice for arbitrary collections of finite sets).

Moreover, by a theorem of Solovay [En-5], if $ZF$ is consistent, then the axiom of choice is independent of $ZF + LM$.

- Our blanket assumption throughout the paper is that *$ZF$ is a consistent first order theory.* Also, a *completion* of a consistent theory $T_0$ refers to a *consistent* completion of $T_0$.

## 2. COUNTABLE LEIBNIZIAN MODELS

This section is devoted to the proof of Theorem 2.1 which refines Mycielski's Theorem 1 mentioned in the introduction.

**Theorem 2.1**. *The following are equivalent for a completion $T$ of $ZF$:*

(i) *$T$ has a Leibnizian model.*

(ii) *$T$ proves $LM$.*

(iii) *$T$ has $2^{\aleph_0}$ nonisomorphic countable Leibnizian models.*

**Proof: (i) $\Rightarrow$ (ii)** is an immediate consequence of the following lemma:

**Lemma 2.1.1.** *If $\mathfrak{M}$ is a model of $ZF+\neg LM$, then the expanded model $(\mathfrak{M},\alpha)_{\alpha\in\mathbf{Ord}}$ contains a pair of indiscernibles.*

**Proof:** Suppose $\mathfrak{M}$ *is a model of* $ZF+\neg LM$. Since $LM$ fails in $\mathfrak{M}$ there are elements $a$ and $b$ of $\mathfrak{M}$ such that

(1) $\qquad \mathfrak{M}\models\forall\gamma>\max\{\rho(a),\rho(b)\}\ Th(V_\gamma,\in,a)=Th(V_\gamma,\in,b).$

We claim that $a$ and $b$ are indiscernible over $(\mathfrak{M},\alpha)_{\alpha\in\mathbf{Ord}^{\mathfrak{M}}}$. To see this, suppose that for some $\alpha_1,\cdots,\alpha_n$ in $\mathbf{Ord}^{\mathfrak{M}}$,

(2) $\qquad \mathfrak{M}\models\varphi(a,\alpha_1,\cdots,\alpha_n).$

By the *extended reflection theorem* of Myhill and Scott [MS, p. 273], there is an ordinal $\theta$ of $\mathfrak{M}$ such that

(3) $\qquad \mathfrak{M}\models$ "$\{a,b\}\subseteq V_\theta$ and $\alpha_1,\cdots,\alpha_n$ are all definable in $(V_\theta,\in)$", and

(4) $\qquad \mathfrak{M}\models\forall x_0\forall x_1\cdots\forall x_n\in V_\alpha\ (\varphi(x_0,x_1,\cdots,x_n)\longleftrightarrow\varphi^{V_\theta}(x_0,x_1,\cdots,x_n)).$

By setting $\gamma=\theta$ in (1), and using (2), (3), and (4), we conclude

(5) $\qquad \mathfrak{M}\models\varphi(b,\alpha_1,\cdots,\alpha_n).$

$\square$ (Lemma 2.1.1)

Now we establish **(ii) $\Rightarrow$ (iii)**. Suppose $T$ proves $LM$. We distinguish the following two cases:

**(a)** $T \vdash \mathbf{V} = \mathbf{OD}$;

**(b)** $T \vdash \mathbf{V} \neq \mathbf{OD}$.

**Proof of (ii) $\Rightarrow$ (iii) for Case (a):** the desired family of size continuum of Leibnizian models of $T$ in this case is the family of (isomorphism types of) minimal e.e.e.'s of the pointwise definable model $\mathfrak{M}_T$ of $T$. To verify this, we need three preliminary lemmas.

**Lemma 2.1.2.** *Every pointwise definable model of* $ZF + \mathbf{V} = \mathbf{OD}$ *has continuum-many countable nonisomorphic minimal e.e.e.'s.*

**Proof:** This follows from [En-4, Lemma 3.1.3]. Alternatively, it is also an immediate consequence of Theorem 3.2 of this paper. □ (Lemma 2.1.2)

Thanks to Lemma 2.1.2, the proof of **(ii)** $\Rightarrow$ **(iii)** for case (a) will be complete once we show minimal e.e.e.'s of $\mathfrak{M}_T$ are Leibnizian. To accomplish this task we need the following result, whose proof is inspired by a parity argument of Ehrenfeucht [Eh] and Gaifman [G-2, Theorem 4.1] involving models of arithmetic.

**Lemma 2.1.3**. *Suppose* $\mathfrak{M} \models ZF$. *If* $\alpha$ *and* $\beta$ *are in* $\mathbf{Ord}^{\mathfrak{M}}$ *and satisfy conditions* (a) *and* (b) *below, then* $\alpha$ *and* $\beta$ *are discernible in* $\mathfrak{M}$.

(a) $\mathfrak{M} \models \alpha > \beta$.

(b) *For some parameter-free definable term* $\tau(x)$ *of* $\mathfrak{M}$, $\mathfrak{M} \models \tau(\alpha) = \beta$.

**Proof:** Suppose, on the contrary, that $\alpha$ and $\beta$ are indiscernible in $\mathfrak{M}$. (a) and (b) together yield

(1) $\qquad \mathfrak{M} \models \tau(\alpha) = \beta < \alpha.$

Arguing in $\mathfrak{M}$, define a class function $\mathbf{F} : \mathbf{Ord} \to \omega$ assigning a natural number $\mathbf{F}(\theta)$ to each ordinal $\theta$ that measures *the length of the longest decreasing sequence of iterates of* $\tau$ *which starts with* $\theta$, i.e., $\mathbf{F}(\theta) = n$ iff

$$\theta > \tau(\theta) > \tau(\tau(\theta)) > \cdots > \tau^n(\theta), \text{and } \tau^{n+1}(\theta) \notin \mathbf{Ord},$$

where $\tau^n$ is the $n$-th iterate of $\tau$. Officially speaking, $\mathbf{F}(\theta) = n$ can be expressed by the following formula $\psi(\theta, n)$ :

$$\exists(\delta_0, \cdots, \delta_{n+1}) \in \mathbf{Ord}^{<\omega}[\delta_0 = \theta \wedge (\forall i < n+1\ \tau(\delta_i) = \delta_{i+1} < \delta_i) \wedge \tau(\delta_{n+1}) \notin \mathbf{Ord}].$$

By (1), $\mathbf{F}(\alpha) = \mathbf{F}(\beta) + 1$ holds within $\mathfrak{M}$. Therefore

(2) $\mathfrak{M} \models$ "$\mathbf{F}(\alpha)$ is even iff $\mathbf{F}(\beta)$ is odd",

This contradicts the indiscernibility of $\alpha$ and $\beta$ in $\mathfrak{M}$. $\square$ (Lemma 2.1.3)

We now use Lemma 2.1.3 to show that minimal e.e.e.'s of $\mathfrak{M}_T$ are Leibnizian.

**Lemma 2.1.4.** *Suppose $T$ is a completion of $ZF + \mathbf{V} = \mathbf{OD}$. If $\mathfrak{M}$ is a minimal elementary extension of $\mathfrak{M}_T$, then $\mathfrak{M}$ is Leibnizian.*

**Proof:** Since $\mathbf{V} = \mathbf{OD}$ holds in $\mathfrak{M}$ it suffices to verify that there are no indiscernibles $\alpha > \beta$ in $\mathbf{Ord}^{\mathfrak{M}}$. Furthermore, we can assume that neither $\alpha$ nor $\beta$ is in $\mathfrak{M}_0$ because every element of $\mathfrak{M}_T$ is definable without parameters. Therefore $\alpha$ and $\beta$ are discernible by Lemma 2.1.3. because condition (b) of Lemma 2.1.3 is satisfied by the minimality of $\mathfrak{M}$ over $\mathfrak{M}_T$. $\square$ (Lemma 2.1.4)

This concludes the proof of (**ii**) $\Rightarrow$ (**iii**) for case (a).

**Proof of Case (b) of** (**ii**) $\Rightarrow$ (**iii**): This case is settled by combining Lemma 2.1.5 and Lemma 2.1.6 below.

**Lemma 2.1.5.** *If $\mathfrak{M}$ is a model of $ZF + LM$ whose ordinals are definable, then $\mathfrak{M}$ is Leibnizian.*

**Proof:** Suppose $\mathfrak{M}$ is a model of $ZF + LM$ whose ordinals are definable. Given distinct $a$ and $b$, we know that

$$(\mathfrak{M}, a, b) \models \exists \alpha > \max\{\rho(a), \rho(b)\}\ Th(V_\alpha, \in, a) \neq Th(V_\alpha, \in, b).$$

Therefore for some $\varphi$,

$$\mathfrak{M} \models [(V_\alpha, \in, a) \models \varphi(a)\ ] \text{ and } [(V_\alpha, \in, b) \models \neg\varphi(b)].$$

Since $\omega^{\mathfrak{M}}$ might be nonstandard, $\varphi$ might be a formula of nonstandard length, however since every ordinal of $\mathfrak{M}$ is definable, both $\varphi$ and $\alpha$ have first order definitions in $\mathfrak{M}$. Hence, there is a first order formula $\psi(x)$ with no parameters from $\mathfrak{M}$, $s$ such that

$$\mathfrak{M} \models \forall x(\psi(x) \leftrightarrow [(V_\alpha, \in, x) \models \varphi(x)]).$$

Therefore $a$ and $b$ are discernible in $\mathfrak{M}$ because $\mathfrak{M} \models \psi(a)$, but $\mathfrak{M} \models \neg\psi(b)$. $\square$ (Lemma 2.1.5)

Lemma 2.1.6 below extends a result of Paris [P] who showed that every completion of $ZF + \mathbf{V} \neq \mathbf{OD}$ has at least two nonisomorphic countable models whose ordinals are definable.

**Lemma 2.1.6.** [En-6] *Every theory extending $ZF + \mathbf{V} \neq \mathbf{OD}$ has $2^{\aleph_0}$ nonisomorphic countable models whose ordinals are definable.*

This concludes the proof of **(ii)** $\Rightarrow$ **(iii)** for case (b). Since **(iii)** $\Rightarrow$ **(i)** is trivial, the proof of Theorem 2.1 is complete. $\square$ (Lemma 2.1.6)

## 3. UNCOUNTABLE LEIBNIZIAN MODELS

In this section we build two families, $\mathcal{F}(T, \kappa)$ and $\mathcal{G}(T)$, of power $2^{\aleph_1}$ of nonisomorphic Leibnizian models of power $\aleph_1$ of a given completion $T$ of $ZF + \mathbf{V} = \mathbf{OD}$, where $\kappa$ is a prescribed infinite cardinal of $\mathfrak{M}_T$ (the pointwise definable model of $T$), such that:

- For every $\mathfrak{M} \in \mathcal{F}(T, \kappa)$, $(\kappa^+)^{\mathfrak{M}}$ is $\aleph_1$-like.
- Every $\mathfrak{M} \in \mathcal{G}(T), \mathbf{Ord}^{\mathfrak{M}}$ is $\aleph_1$-like.

Recall that a linear order $(X, <)$ is said to be $\theta$-like, where $\theta$ is an infinite cardinal, if $|X| = \theta$ but the set of $<$-predecessors of each element of $X$ has cardinality less than $\theta$.

### 3.1. The Construction of $\mathcal{F}(T, \kappa)$

**Theorem 3.1.** *Suppose $T$ is a completion of $ZF + \mathbf{V} = \mathbf{OD}$, and let $\kappa$ be a prescribed infinite cardinal of $\mathfrak{M}_T$. There exist $2^{\aleph_1}$ nonisomorphic Leibnizian models $\mathfrak{N}$ of $T$ such that $(\kappa^+)^{\mathfrak{M}}$ is $\aleph_1$-like.*

We need to prove a series of preliminary lemmas.

**Lemma 3.1.1.** *Suppose $\mathfrak{M}$ is a countable model of $ZFC$ and $\kappa$ is a regular cardinal of $\mathfrak{M}$. There exist minimal $\kappa$-e.e.e.'s $\mathfrak{N}_1$ and $\mathfrak{N}_2$ of $\mathfrak{M}$ such that $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are nonisomorphic over $\mathfrak{M}$.*

**Proof:** By [En-1, Theorem 2.12] there exist minimal $\kappa$-e.e.e.'s $\mathfrak{N}_1$ and $\mathfrak{N}_2$ such that:

(a) $\kappa^{\mathfrak{N}_1} \backslash \kappa^{\mathfrak{M}}$ has a least member.

(b) $\kappa^{\mathfrak{N}_2} \backslash \kappa^{\mathfrak{M}}$ has no least member.

This shows that $\mathfrak{N}_1$ *and* $\mathfrak{N}_2$ are nonisomorphic over $\mathfrak{M}$[3]. $\square$ (Lemma 3.1.1)

In what follows $2^\alpha$ denotes the set of binary sequences of length $\alpha$, where $\alpha$ is an ordinal.

**Lemma 3.1.2.** *Suppose $T$ is a completion of $ZF + \mathbf{V} = \mathbf{OD}$, and $\kappa$ is an infinite cardinal of $\mathfrak{M}_T$. There is a family $\mathcal{F}(T,\kappa) = \{\mathfrak{M}_s : s \in 2^{\omega_1}\}$ of nonisomorphic elementary extensions of $\mathfrak{M}_T$ such that for every $s \in 2^{\omega_1}$*

(a) *$\mathfrak{M}_s$ fixes every element of $\kappa^{\mathfrak{M}_T}$;*

(b) *$(\kappa^+)^{\mathfrak{M}_s}$ is $\aleph_1$-like;*

(c) *$\mathfrak{M}_s$ is generated from the elements of $(\kappa^+)^{\mathfrak{M}_s}$ (via the definable terms).*

**Proof:** It is routine, using Lemma 3.1.1, to build a family of models $\{\mathfrak{M}_s : s \in \bigcup\limits_{\alpha \leq \omega_1} 2^\alpha\}$ by recursion on the length of $s$, such that

(1) $\mathfrak{M}_\emptyset = \mathfrak{M}_0$;

(2) For every $s \in \bigcup\limits_{\alpha<\omega_1} 2^\alpha$, $\mathfrak{M}_{s\frown 0}$ and $\mathfrak{M}_{s\frown 1}$ are minimal $\kappa^+$-e.e.e.'s of $\mathfrak{M}_s$ which are nonisomorphic over $\mathfrak{M}_s$;

(3) For every limit $\alpha \leq \omega_1$, and every $s \in 2^\alpha$, $\mathfrak{M}_s = \bigcup\limits_{\beta<\alpha} \mathfrak{M}_{s\restriction\beta}$.

It is also routine to show that for every $s \in 2^{\omega_1}$, conditions (a) and (b) of the lemma are satisfied. To see that condition (c) is satisfied, for each $\alpha < \omega_1$ choose

$$\delta_\alpha \in (\kappa^+)^{\mathfrak{M}_{s(\alpha+1)}} \backslash (\kappa^+)^{\mathfrak{M}_{s(\alpha)}},$$

and use the fact that $\mathfrak{M}_{s(\alpha+1)}$ is a minimal elementary extension of $\mathfrak{M}_{s(\alpha)}$ to verify that $\mathfrak{M}_s$ is generated from $\{\delta_\alpha : \alpha < \omega_1\}$. It remains to show that if $s$ and $t$ are distinct elements of $2^{\omega_1}$, then $\mathfrak{M}_s$ and $\mathfrak{M}_t$ are nonisomorphic. Given distinct $s$ and $t$ in $2^{\omega_1}$, let

$$\theta := \min\{\alpha < \omega_1 : s(\alpha) \neq t(\alpha)\}.$$

Therefore by (2),

[3] With a little more work one can show that there are continuum-many countable $\kappa$-e.e.e.'s of $\mathfrak{M}$ which are pairwise nonisomorphic over $\mathfrak{M}$.

(4) $\mathfrak{M}_{s\restriction\theta+1}$ is not isomorphic to $\mathfrak{M}_{t\restriction\theta+1}$ over $\mathfrak{M}_{s\restriction\theta}$ $(= \mathfrak{M}_{t\restriction\theta})$.

On the other hand, the construction of $\mathfrak{M}_s$ makes it evident that the family $\{\mathfrak{N} : \mathfrak{N} \prec \mathfrak{M}_s\}$ of elementary submodels of $\mathfrak{M}_s$ has order type $\omega_1$ under $\prec$ . Therefore for every $\alpha < \omega_1$ $\mathfrak{M}_{s\restriction\alpha}$ is unambiguously recoverable from $\mathfrak{M}_s$ since

(5) $\mathfrak{M}_{s\restriction\alpha}$ is the $\alpha$-th elementary submodel of $\mathfrak{M}_s$.

(4) and (5) together show that

(6) $\mathfrak{M}_{s\restriction\theta+1}$ is not isomorphic to $\mathfrak{M}_{t\restriction\theta+1}$.

This makes it clear that $\mathfrak{M}_s \ncong \mathfrak{M}_t$, as desired. □ (Lemma 3.1.2)

**Lemma 3.1.3.** (Solovay, private communication) *Suppose* $\mathfrak{M}$ *is a model of* $ZF + \mathbf{V} = \mathbf{OD}$*, and* $\kappa$ *is an ordinal of* $\mathfrak{M}$ *all of whose elements are definable in* $\mathfrak{M}$*. If* $\mathfrak{N}$ *is the model generated by the elements of the* $\mathfrak{M}$*-power set* $\mathcal{P}^{\mathfrak{M}}(\kappa)$ *via the definable terms of* $\mathfrak{M}$*, then* $\mathfrak{N}$ *is Leibnizian.*

**Proof:** Suppose, on the contrary, that

(1) $a$ and $b$ form a pair of indiscernibles in $\mathfrak{N}$.

By the choice of $\mathfrak{N}$, there is a definable term $\tau(x, y)$ such that $a = \tau^{\mathfrak{M}}(U)$ for some $U \in \mathcal{P}^{\mathfrak{M}}(\kappa)$ (thanks to the availability of a definable pairing function, finitely many parameters can be coded into one). Coupled with (1), this implies

(2) $b = \tau^{\mathfrak{M}}(V)$ for some $V \in \mathcal{P}^{\mathfrak{M}}(\kappa)$.

Let $\lhd$ denote the parameter-free definable global well-ordering in $\mathfrak{N}$. Define $U_0$ and $V_0$ as

(3) $U_0$ is the $\lhd$-first $U$ such that $a = \tau(U)$, and $V_0$ is the $\lhd$-first $V$ such that $b = \tau(V)$.

(1) and (3) together imply

(4) $U_0$ and $V_0$ form a pair of indiscernibles in $\mathfrak{N}$.

But since every element of $\kappa$ is definable, and $U_0$ and $V_0$ are both subsets of $\kappa^{\mathfrak{M}}$, (4) implies that $U_0 = V_0$ (if $U_0 \neq V_0$, there is a definable element in the symmetric difference $U_0 \triangle V_0$, contradicting (4)). Hence $a = \tau(U_0) = \tau(V_0) = b$, contradiction. $\square$ (Lemma 3.1.3)

**Proof of Theorem 3.1:** Given a completion $T$ of $ZF + \mathbf{V} = \mathbf{OD}$, let $\mathcal{F}(T, \kappa)$ be the family given by Lemma 3.1.2. In light of Lemmas 3.1.2 and 3.1.3, it suffices to verify that every model $\mathfrak{M}$ in $\mathcal{F}(T, \kappa)$ is generated by the elements of $\mathcal{P}^{\mathfrak{M}}(\kappa)$. Since $\mathfrak{M}$ satisfies condition (c) of Lemma 3.1.2, $\mathfrak{M}$ is generated from the elements of $(\kappa^+)^{\mathfrak{M}_s}$. Therefore the proof would be complete once we verify that there is a *parameter-free definable injective map* $f$ such that

$$T \vdash f : \kappa^+ \to \mathcal{P}(\kappa).$$

Let $\lhd$ be a parameter-free definable global well-order in $\mathfrak{M}$, and let $\Gamma$ be a parameter-free definable bijection between $\mathbf{Ord} \times \mathbf{Ord}$ and $\mathbf{Ord}$ [J, p. 20]. For $\kappa \leq \alpha < \kappa^+$, let $S_\alpha$ be the $\lhd$-first subset $S$ of $\kappa \times \kappa$ such that $(\alpha, \in) \cong (\kappa, S)$, and define

$$f(\alpha) := \begin{cases} \{\alpha\}, & \text{if } \alpha < \kappa, \\ \{\Gamma(x,y) : (x,y) \in S_\alpha\}, & \text{if } \kappa \leq \alpha < \kappa^+ . \end{cases}$$

$\square$ (Theorem 3.1)

**Remark 3.1.4.** Let $KM$ be the Kelley- Morse theory of classes. The strategy of the proof of Theorem 3.1 can also be used to prove that every completion of the $KM$ + $\Pi^1_\infty$-*CHOICE*[4] has $2^{\aleph_1}$ nonisomorphic Leibnizian models $(\mathfrak{M}, \mathcal{A})$, where $\mathfrak{M} = (M, E)$ is a model of $ZFC$ and forms the "sets" of the model, and $\mathcal{A} \subseteq \mathcal{P}(M)$ forms the "classes" of $\mathfrak{M}$, such that $M$ is countable and $\mathcal{A}$ has power $\aleph_1$. A similar comment applies to completions of the theory $Z_2$+ $\Pi^1_\infty$-*CHOICE*, where $Z_2$ is "second order number theory".

**Remark 3.1.5.** Suppose $\kappa$ is a definable weakly compact cardinal of a completion $T$ of $ZF + \mathbf{V} = \mathbf{OD}$ (where $\kappa = \aleph_0$ is allowed). As shown in [En-7], there is a family of size $2^{2^{\aleph_0}}$ of nonisomorphic Leibnizian models $\mathfrak{M}$ of $T$ of power $2^{\aleph_0}$ such that $\kappa^{\mathfrak{M}}$ is $2^{\aleph_0}$-like.

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[4] $\Pi^1_\infty$-*CHOICE* is the scheme in the theory of classes consisting of all sentences of the form

$$\forall \alpha \in \mathbf{Ord}\ \exists X \subseteq \mathbf{Ord}\ \varphi(\alpha, X) \to \exists Y \subseteq \mathbf{Ord}\ \forall \alpha \in \mathbf{Ord}\ \varphi(\alpha, (Y)_\alpha).$$

Here $(Y)_\alpha = \{y : \Gamma(\alpha, y) \in Y\}$, where $\Gamma$ is a canonical pairing function.

### 3.2. The Construction of $\mathcal{G}(T)$

We now turn to the construction of a family $\mathcal{G}(T)$ of size $2^{\aleph_1}$ of nonisomorphic $\aleph_1$-like Leibnizian models of a completion $T$ of $ZF + \mathbf{V} = \mathbf{OD}$. Recall that a model $\mathfrak{M} = (M, E)$ of $ZF$ is said to be *$\kappa$-like*, where $\kappa$ is an uncountable cardinal, if

$$|M| = \kappa \text{ but } \forall a \in M, |\{x \in M : xEa\}| < \kappa.$$

It is easy to see that if $\mathfrak{M}$ is a model of $ZFC$, then $\mathfrak{M}$ is $\kappa$-like iff its linearly ordered set of ordinals $\mathbf{Ord}^{\mathfrak{M}}$ is $\kappa$-like. The axiom of choice is important here since we can start with an $\aleph_1$-like model $\mathfrak{M}$ of $ZF$ and add continuum-many mutually generic Cohen reals to $\mathfrak{M}$ to obtain a model $\mathfrak{N}$, in which the axiom of choice fails, and whose class of ordinals is $\aleph_1$-like, but which is itself not $\aleph_1$-like. The classical theorem of Keisler and Morley on e.e.e.'s of models of $ZF$ [CK, theorem 2.2.18] implies - via an iteration of length $\omega_1$ - that every completion $T$ of $ZF$ has an $\aleph_1$-like model. Keisler's work [Ke, Section 5] also shows that every completion of $ZF$ has $2^{\aleph_1}$ nonisomorphic $\aleph_1$-like models. However, as shown in [En-2], for $\kappa \geq \aleph_2$ the existence of $\kappa$-like models of a completion $T$ of $ZFC$, depends both on the object theory $T$, as well as the meta-theory adopted. See also Remark 3.3.2 (a,b) below.

We need a new idea - superminimality - to build $\aleph_1$-like Leibnizian models:

- An e.e.e. $\mathfrak{N}$ of $\mathfrak{M}$ is a *superminimal* extension, if for any pair of elements $a \in N \backslash M$, and $b \in N$ there is a definable term $\tau(x)$ *without parameters* such that $\mathfrak{M} \models \tau(a) = b$.

**Theorem 3.2**[5] *Every countable model $\mathfrak{M}$ of $ZF + \mathbf{V} = \mathbf{OD}$ has a superminimal e.e.e. . Moreover, there are continuum-many countable superminimal e.e.e.'s of $\mathfrak{M}$ which are pairwise nonisomorphic over $\mathfrak{M}$.*

**Proof:** Let $\mathfrak{M} = (M, E)$ be a countable model of $ZF + \mathbf{V} = \mathbf{OD}$, $(\mathbf{V}^{\mathbf{Ord}})^{\mathfrak{M}}$ be the family of functions from $\mathbf{Ord}^{\mathfrak{M}}$ into $M$ whose graphs are parametrically definable in $\mathfrak{M}$, and $\mathbb{B}$ be the Boolean algebra of subsets of $\mathbf{Ord}^{\mathfrak{M}}$ which are parametrically definable in $\mathfrak{M}$. Given an ultrafilter $\mathcal{U}$ on $\mathbb{B}$, and $f \in (\mathbf{V}^{\mathbf{Ord}})^{\mathfrak{M}}$,

[5] A similar proof shows that *every countable model of Peano arithmetic has a superminimal e.e.e.* This result was independently obtained by Kossak and Schmerl [KS], and generalizes a result of Knight [Kn].

$[f]_{\mathcal{U}}$ as usual is the $\mathcal{U}$-equivalence class of $f$ consisting of members of $(\mathbf{V}^{\mathbf{Ord}})^{\mathfrak{M}}$ which agree with $f$ on a member of $\mathcal{U}$. Let $\mathfrak{N}_{\mathcal{U}} = (N_{\mathcal{U}}, F_{\mathcal{U}})$, where

$$N_{\mathcal{U}} := (\mathbf{V}^{\mathbf{Ord}})^{\mathfrak{M}}/\mathcal{U} := \{[f]_{\mathcal{U}} : f \in (\mathbf{V}^{\mathbf{Ord}})^{\mathfrak{M}}\},$$

and

$$[f]_{\mathcal{U}}\ F_{\mathcal{U}}\ [g]_{\mathcal{U}} \text{ iff } \{\alpha \in \mathbf{Ord}^{\mathfrak{M}} : f(\alpha)\ E\ g(\alpha)\} \in \mathcal{U}.$$

Thanks to the presence of a definable global well-ordering within $\mathfrak{M}$, the Łoš Theorem for ultrapowers holds in this context. Consequently, if $\mathcal{U}$ is a non-principal ultrafilter, then $\mathfrak{N}_{\mathcal{U}}$ is a proper elementary extension of $\mathfrak{M}$ (with the obvious identification of the $\mathcal{U}$-equivalence classes of constant maps with elements of $\mathfrak{M}$).

We now describe the construction of an ultrafilter $\mathcal{U}$ on $\mathbb{B}$ such that $\mathfrak{N}_{\mathcal{U}}$ is a superminimal e.e.e. of $\mathfrak{M}$. Our construction involves weaving an external recursion of length $\omega$ with internal recursions of length $\mathbf{Ord}^{\mathfrak{M}}$. Let

$$((f_n, \alpha_n) : n \in \omega)$$

be an enumeration of the Cartesian product $(\mathbf{V}^{\mathbf{Ord}})^{\mathfrak{M}} \times \mathbf{Ord}^{\mathfrak{M}}$. We shall recursively construct a sequence $(X_n : n \in \omega)$ of elements of $\mathbb{B}$, and a sequence of *parameter-free* definable terms $(\tau_n : n \in \omega)$, such that (1) through (3) below hold:

(1) $\quad \forall n \in \omega\ (X_n \supseteq X_{n+1})$ and $X_n$ is unbounded in $\mathbf{Ord}^{\mathfrak{M}}$;

(2) $\quad \forall n \in \omega\ (f_n \restriction X_{n+1}$ is constant, or $f_n \restriction X_{n+1}$ is injective);

(3) $\quad \forall n \in \omega\ (f_n \restriction X_{n+1}$ is injective $\Rightarrow \forall x \in X_{n+1}(\tau_n^{\mathfrak{M}}(f_n(x)) = \alpha_n)$.

At stage 0 of the construction, we set $X_0 = \mathbf{Ord}^{\mathfrak{M}}$, and at stage $(n+1)$, we define $\tau_n$ and $X_{n+1}$ from $X_n$ by distinguishing two cases:

I. $f_n \restriction X_n$ is a set.

II. $f_n \restriction X_n$ is a proper class.

If case I holds, then by the replacement scheme, there exists $\theta \in \mathbf{Ord}^{\mathfrak{M}}$ such that $f_n^{-1}(\theta) \cap X_n$ is unbounded in $\mathbf{Ord}^{\mathfrak{M}}$. So define $X_{n+1} := f_n^{-1}(\theta_0) \cap X_n$, where $\theta_0 =$ least $\theta \in \mathbf{Ord}^{\mathfrak{M}}$ such that $f_n^{-1}(\theta) \cap X_n$ is unbounded in $\mathbf{Ord}^{\mathfrak{M}}$. In this case define $\tau_n(x)$ to be identically zero.

If case II holds then first construct an unbounded $Y \subseteq X_n$ such that $f \restriction Y$ is injective and $Y \in \mathbb{B}$ by transfinite, using transfinite recursion within $\mathfrak{M}$:

- $y_0 =$ the first member of $X_n$.
- $y_\alpha =$ the least member $x$ of $X_n$ such that $\forall \gamma < \alpha,\ f_n(x) > f_n(y_\gamma)$.

Note that $y_\alpha$ is well-defined for all ordinals $\alpha$ of $\mathfrak{M}$ since $X_n$ is a proper class, and $\{f_n(y_\gamma) : \gamma < \alpha\}$ is a set (thanks to the replacement scheme). Let $\psi(x, \delta_0)$ be a formula which defines $\{f_n(y) : y \in Y\}$ in $\mathfrak{M}$, where $\delta_0$ is a parameter in $\mathbf{Ord}^{\mathfrak{M}}$. For each ordinal $\delta$, let $C_\delta$ be the class $\{x : \psi(x, \delta)\}$. Within $\mathfrak{M}$ we wish to define a uniform refinement $D_\delta \subseteq C_\delta$ for each $\delta \in \mathbf{Ord}$ such that conditions A and B below hold:

A. $\forall \delta \in \mathbf{Ord}$ ($C_\delta$ is a proper class $\Rightarrow D_\delta$ is a proper class);

B. $\forall \delta, \delta' \in \mathbf{Ord}$ $(\delta \neq \delta' \Rightarrow D_\delta \cap D_{\delta'} = \emptyset)$.

Since $C_{\delta_0}$ is a proper class, $\{\Gamma(x, \delta) : x \in C_\delta,\ \delta \in \mathbf{Ord}\}$ is also a proper class, and can be enumerated by a sequence $(a_\alpha : \alpha \in \mathbf{Ord})$, where $\Gamma$ is a parameter-free definable bijection between $\mathbf{Ord} \times \mathbf{Ord}$ and $\mathbf{Ord}$. Also, for each ordinal $\theta$ let $(\theta)_1$ and $(\theta)_2$ be the unique ordinals satisfying

$$\Gamma((\theta)_1, (\theta)_2) = \theta.$$

Define a subsequence $(b_\alpha : \alpha \in \mathbf{Ord})$ of $(a_\alpha : \alpha \in \mathbf{Ord})$ by transfinite recursion which is *monotone increasing in its first coordinate* as follows:

- $b_0 := a_0$;
- $b_\alpha :=$ the first $a_\theta$ such that for all $\gamma < \alpha$, $(a_\theta)_1 > (b_\gamma)_1$.

Next, define

$$D_\delta := C_\delta \cap \{(b_\alpha)_1 : \alpha \in \mathbf{Ord})\}.$$

It is routine to verify that the $D_\delta$'s are well-defined and satisfy conditions A and B. We are now in a position to describe a term $\sigma(x)$ which will serve as $\tau_n(x)$:

$$\sigma(x) := \begin{cases} (\theta)_1, & \text{if } \exists \delta (y \in D_\delta, \text{ and } x \text{ is the } \theta\text{-th element of } D_\delta). \\ 0, & \text{otherwise.} \end{cases}$$

It is now easy to see, using condition A, that

(4) $\forall \alpha, \delta \in \mathbf{Ord}$ ($D_\delta$ is a proper class $\Rightarrow \sigma^{-1}(\alpha) \cap D_\delta$ is a proper class).

Finally, we define $X_{n+1}$ by:

$$X_{n+1} := \{x \in \mathbf{Ord}^{\mathfrak{M}} : \mathfrak{M} \models (x \in D_{\delta_0} \wedge \sigma(x) = \alpha_n)\}.$$

By (4) $X_{n+1}$ is an $\mathfrak{M}$-definable unbounded subset of $\mathbf{Ord}^{\mathfrak{M}}$, and

$$\forall x \in X_{n+1}(\sigma^{\mathfrak{M}}(f_n(x)) = \alpha_n).$$

This concludes the description of the sequences $(X_n : n \in \omega)$ and $(\tau_n : n \in \omega)$ satisfying conditions (1) through (3).

The family of $X_n$'s forms a pre-filter by (1), and by (2) the characteristic function $\chi_S : \mathbf{Ord}^{\mathfrak{M}} \to \{0, 1\}$ of every $S \in \mathbb{B}$ is constant on some $X_n$. Therefore, since every $X_n$ is unbounded in $\mathbf{Ord}^{\mathfrak{M}}$ by (1), the pre-filter $\{X_n : n \in \omega\}$ can be *uniquely* extended to a *nonprincipal* ultrafilter $\mathcal{U}$ on $\mathbb{B}$.

The ultrapower $\mathfrak{N}_{\mathcal{U}}$ is an e.e.e. of $\mathfrak{M}$ since as noted earlier, the Łoš Theorem on ultrapowers holds in this context, and by (2) every bounded function in $(\mathbf{V}^{\mathbf{Ord}})^{\mathfrak{M}}$ is constant on a member of $\mathcal{U}$. We now arrive at the crucial part of the proof, where we verify that $\mathfrak{N}$ is a *superminimal* extension of $\mathfrak{M}$. We shall first show that $\mathfrak{N}$ is a *minimal* extension of $\mathfrak{M}$. Observe that if the graph of $f \in (\mathbf{V}^{\mathbf{Ord}})^{\mathfrak{M}}$ is defined by some formula $\varphi(x, y, \delta_0)$ ($\delta_0$ is a parameter in $\mathbf{Ord}^{\mathfrak{M}}$), then

$$\mathfrak{N}_{\mathcal{U}} \models \varphi([id]_{\mathcal{U}}, [f]_{\mathcal{U}}, \delta_0),$$

where $id$ is the identity map on $\mathbf{Ord}^{\mathfrak{M}}$. Given $[f]_{\mathcal{U}} \in N_{\mathcal{U}} \backslash M$, $f = f_n$ for some $n \in \omega$, so by (2) $f \restriction X_{n+1}$ is injective. Let $\psi(x, \delta_1)$ define $X_{n+1}$ in $\mathfrak{M}$. Then

$$\mathfrak{N}_{\mathcal{U}} \models \psi([id]_{\mathcal{U}}, \delta_1),$$

since $X_{n+1} \in \mathcal{U}$. Moreover

$$\mathfrak{N}_{\mathcal{U}} \models \forall x \forall x' \forall y \ [\psi(x, \delta_1) \wedge \psi(x', \delta_1) \wedge \varphi(x, y, \delta_0) \wedge \varphi(x', y, \delta_0)] \to x = x',$$

because $\mathfrak{M} \prec \mathfrak{N}_{\mathcal{U}}$. Therefore $[id]_{\mathcal{U}}$ can be defined in $\mathfrak{N}_{\mathcal{U}}$ as the *unique* element $x$ satisfying the formula

$$\varphi(x, [f]_{\mathcal{U}}, \delta_0) \wedge \psi(x, \delta_1).$$

So now we know that $\mathfrak{N}_{\mathcal{U}}$ is a minimal elementary extension of $\mathfrak{M}$.

Since $\mathfrak{N}_{\mathcal{U}}$ is a minimal extension of $\mathfrak{M}$ and $\mathbf{V} = \mathbf{OD}$ holds in $\mathfrak{M}$, $\mathfrak{N}$ is shown to be a superminimal extension of $\mathfrak{M}$ if every $\alpha \in \mathbf{Ord}^{\mathfrak{M}}$ is definable in $\mathfrak{N}$ from every element of $N_{\mathcal{U}}\backslash M$. So choose $[f]_{\mathcal{U}} \in N_{\mathcal{U}}\backslash M$, and $\alpha \in \mathbf{Ord}^{\mathfrak{M}}$. Then for some $n \in \omega$, $(f, \alpha) = (f_n, \alpha_n)$. Therefore by (2) and (3):

$$\mathfrak{N}_{\mathcal{U}} \models \tau_n([f]_{\mathcal{U}}) = \alpha.$$

This concludes the construction of a superminimal e.e.e. of $\mathfrak{M}$.

We now verify that indeed there are *continuum many* superminimal e.e.e.'s of $\mathfrak{M}$ which are pairwise nonisomorphic over $\mathfrak{M}$. The basic idea is to modify the construction of the ultrafilter $\mathcal{U}$ by a construction which produces a family of ultrafilters $\{\mathcal{U}_g : g \in 2^\omega\}$. To do so, we need a uniform way of splitting definable classes $X$ in $\mathbb{B}$ into two disjoint pieces. This is easy: if $X \in \mathbb{B}$ is unbounded in $\mathbf{Ord}^{\mathfrak{M}}$, then $X$ can be enumerated in increasing order within $\mathfrak{M}$ as $\{x_\alpha : \alpha < \mathbf{Ord}^{\mathfrak{M}}\}$, therefore $\mathrm{Lim}(X) := \{x_\alpha : \alpha$ is a limit ordinal of $\mathfrak{M}\}$, and $\mathrm{Succ}(X) := \{x_\alpha : \alpha$ is a successor ordinal of $\mathfrak{M}\}$ yield the desired partition. To construct the ultrafilter $\mathcal{U}_h$, given $g : \omega \to \{0, 1\}$, modify the recursive definition of $\mathcal{U}$ as follows: to construct $\tau_n$ and $X^g_{n+1}$ at stage $(n+1)$, first refine $X^g_n$ to $Z^g_n$, where

$$Z^g_n := \begin{cases} Succ(X^g_n), \text{ if } g(n) = 0, \\ Lim(X^g_n), \text{ if } g(n) = 1. \end{cases}$$

Then define $\tau_n$ and $X^g_{n+1} \subseteq Z^g_n$ as before so that (1), (2), and (3) are satisfied. For each $n \in \omega$ let $\varphi^g_n(x, \delta_n)$ define $Z^g_n$ in $\mathfrak{M}$ (where $\delta_n$ is a parameter in $\mathbf{Ord}^{\mathfrak{M}}$). Since $[id]_{\mathcal{U}_g}$ satisfies the 1-type

$$\Gamma^g(x) := \{\varphi^g_n(x, \delta_n) : n \in \omega\},$$

and $\Gamma^g(x)$ and $\Gamma^{\overline{g}}(x)$ are incompatible for $g \neq \overline{g}$, there are continuum many complete 1-types over $\mathfrak{M}$ which are realized in the family of countable superminimal e.e.e.'s of $\mathfrak{M}$. This shows that there are continuum many superminimal e.e.e.'s of $\mathfrak{M}$ which are pairwise nonisomorphic over $\mathfrak{M}$. $\square$ (Theorem 3.2)

**Theorem 3.3.** *Every completion $T$ of $ZF + \mathbf{V} = \mathbf{OD}$ has $2^{\aleph_1}$ nonisomorphic $\aleph_1$-like Leibnizian models.*

**Proof:** We first describe the construction of an $\aleph_1$-like Leibnizian model of a completion $T$ of $ZF + \mathbf{V} = \mathbf{OD}$. Start with a pointwise definable model $\mathfrak{M}_T$ of $T$, and use Theorem 3.2 to build a continuous e.e.e. chain of countable models $(\mathfrak{M}_\alpha : \alpha \leq \omega_1)$ such that for every ordinal $\alpha < \omega_1$, $\mathfrak{M}_{\alpha+1}$ is a superminimal e.e.e. of $\mathfrak{M}_\alpha$. In light of the fact that $\mathfrak{M}_T$ is Leibnizian, the following lemma shows that $\mathfrak{M}_{\omega_1}$ is Leibnizian.

**Lemma 3.3.1.** *If* $(\mathfrak{M}_\alpha : \alpha \leq \omega_1)$ *is a sequence of models satisfying conditions* (a) *through* (c) *below, then* $\mathfrak{M}_{\omega_1}$ *is an* $\aleph_1$*-like Leibnizian e.e.e. of* $\mathfrak{M}_0$.

(a) $\mathfrak{M}_0$ *is Leibnizian model of* $ZF + \mathbf{V} = \mathbf{OD}$;

(b) *For every ordinal* $\alpha < \omega_1$, $\mathfrak{M}_{\alpha+1}$ *is a superminimal e.e.e. of* $\mathfrak{M}_\alpha$;

(c) *For every limit ordinal* $\alpha \leq \omega_1$, $\mathfrak{M}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{M}_\beta$.

**Proof:** It is easy to see that $\mathfrak{M}_{\omega_1}$ is an $\aleph_1$-like e.e.e. of $T$. To show that $\mathfrak{M}_{\omega_1}$ is Leibnizian it suffices to verify that for every ordinal $\alpha < \omega_1$, $\mathfrak{M}_\alpha$ is Leibnizian. This is achieved by induction on $\alpha$ : (1) $\mathfrak{M}_0$ is Leibnizian by assumption; (2) if $\mathfrak{M}_\alpha$ is Leibnizian then by condition (b), Lemma 2.1.3, and the fact that $\mathfrak{M}_{\alpha+1}$ satisfies $\mathbf{V} = \mathbf{OD}$, $\mathfrak{M}_{\alpha+1}$ is also Leibnizian; (3) the case of limit $\alpha$ is easily handled by (c), and Tarski's elementary chain theorem. □(Lemma 3.3.1)

Given a completion $T$ of $ZF + \mathbf{V} = \mathbf{OD}$, use Theorem 3.2 to construct a family of models

$$\mathcal{G}(T) := \{\mathfrak{M}_s : s \in 2^{\leq\omega_1}\},$$

by recursion on the length of $s$, such that

- $\mathfrak{M}_\emptyset = \mathfrak{M}_T$;
- For every $s \in \bigcup_{\alpha<\omega_1} 2^\alpha$, $\mathfrak{M}_{s^\frown 0}$ and $\mathfrak{M}_{s^\frown 1}$ are superminimal e.e.e.'s of $\mathfrak{M}_s$ which are nonisomorphic over $\mathfrak{M}_s$;
- For every limit $\alpha \leq \omega_1$, and every $s \in 2^\alpha$, $\mathfrak{M}_s = \bigcup_{\beta<\alpha} \mathfrak{M}_{s\restriction\beta}$.

By Lemma 3.3.1 $\mathfrak{M}_s$ is an $\aleph_1$-like Leibnizian model of $T$ for every $s \in 2^{\omega_1}$. The verification of the fact that $\mathfrak{M}_s$ and $\mathfrak{M}_t$ are nonisomorphic for distinct $s, t$ in $2^{\omega_1}$ is very similar to the proof of Lemma 3.1.2, and is left to the reader. □ (Theorem 3.3)

**Remark 3.3.2.**

(a) In general, $\aleph_1$ cannot be replaced by $2^{\aleph_0}$ in the statement of Theorem 3.3 because, assuming the consistency of ($ZFC + \exists$ $\omega$-Mahlo cardinal"), it is consistent that many completions of $ZF + \mathbf{V} = \mathbf{OD}$ lack $2^{\aleph_0}$-like models. This is a consequence of [En-3, Theorem 4.9] which shows that if $ZFC+$

"$\exists$ $\omega$-Mahlo cardinal" is consistent, then so is $ZFC + 2^{\aleph_0} = \aleph_2$ + "the only $\aleph_2$-like models of $ZFC$ are those satisfying the recursive scheme $\Phi$", where

$$\Phi := \{\exists\kappa(\kappa \text{ is } n\text{-Mahlo and } V_\kappa \prec_n \mathbf{V}) : n \in \omega\}.$$

(b) Since every consistent extension of $ZFC + \Phi$ has a $\kappa$-like model for all uncountable cardinals $\kappa$ [En-2, Theorem 3.8], one might wonder whether one can prove in $ZFC$ that every consistent extension of the theory $T_0$ obtained by adding $\mathbf{V} = \mathbf{OD}$ $+\Phi$ to $ZF$ has a $2^{\aleph_0}$-like Leibnizian model. Assuming the consistency of $ZFC$ + "$\exists$ weakly compact cardinal", this turns out to be false, since:

(1) By Gödel's second incompleteness theorem, the theory $T_0 + \neg Con(T_0)$ is consistent, but has no $\omega$-standard model;

(2) If $\aleph_2$ has the tree property, then every non $\omega$-standard $\aleph_2$-like model has large sets of indiscernibles [En-3, Theorem 4.5(iiii)];

(3) As shown by Silver [Mi], Con ($ZF + 2^{\aleph_0} = \aleph_2 + \aleph_2$ has the tree property) iff Con ($ZFC$ + "$\exists$ weakly compact cardinal").

(c) Abramson and Harrington [AH, p.593] have asked whether the Shepherdson-Cohen minimal model of set theory [C, III.6] has "large" e.e.e.'s without indiscernibles. The proof of Theorem 3.3 answers this question if "large" is interpreted as "uncountable" since the Leibnizian models produced in the proof of Theorem 3.3 are e.e.e.'s of the pointwise definable model $\mathfrak{M}_T$ of the prescribed theory $T$.

Parts (b) and (c) of Remark 3.3.2 motivate the following questions:

**Question 3.3.3.** Suppose $T$ is a completion of ($ZF + \mathbf{V} = \mathbf{OD}$ $+\Phi$) which has an $\omega$-standard model. Does $T$ have a $2^{\aleph_0}$-like Leibnizian model?

**Question 3.3.4.** Suppose $2^{\aleph_0} > \aleph_1$ and $T$ is a completion of $ZF + \mathbf{V} = \mathbf{OD}$. Does $\mathfrak{M}_T$ have a Leibnizian e.e.e. of power $2^{\aleph_0}$?

**Remark 3.3.5**. Let us define a model $\mathfrak{M}$ to be a *maximal* (*e-maximal*) *Leibnizian* model if $\mathfrak{M}$ is Leibnizian, but no proper elementary (end) extension of $\mathfrak{M}$ is Leibnizian. By Zorn's lemma every Leibnizian model has maximal as well as e-maximal elementary extensions. Since the proof of Theorem 3.3 shows that every countable Leibnizian model of $ZF + \mathbf{V} = \mathbf{OD}$ can be elementarily end extended to an $\aleph_1$-like Leibnizian model, we conclude:

(a) *Every maximal Leibnizian models of* $ZF + \mathbf{V} = \mathbf{OD}$ *is uncountable.*

Moreover, since by [En-1, Theorem 1.5 (b)] no $\aleph_1$-like e.e.e. of a pointwise definable model of $ZF$ has an e.e.e.,

(b) *Every completion of* $ZF + \mathbf{V} = \mathbf{OD}$ *has an* $\aleph_1$*-like e-maximal Leibnizian model.*

With a little more work one can even construct, for every completion of $ZF + \mathbf{V} = \mathbf{OD}$, an e-maximal Leibnizian model which is not a maximal Leibnizian model. These considerations motivate the following questions:

**Question 3.3.6.** Is the theory ($ZFC$+ $2^{\aleph_0} > \aleph_1$ + "some maximal Leibnizian model of $ZF$ has power $\aleph_1$") consistent?

**Question 3.3.7.** Let $\Omega(T)$ denote the set of isomorphism types of maximal Leibnizian models of a completion $T$ of $ZF + LM$. What are the possible values for $|\Omega(T)|$?

**Question 3.3.8.** Is every maximal Leibnizian model of $ZF + LM$ uncountable?

We also take this opportunity to reiterate an open question posed in [AH].

**Question 3.3.9.** (Abramson and Harrington). Does every completion $T$ of $ZF$ have an uncountable model without a pair of indiscernible ordinals?

## 4. WELL-FOUNDED LEIBNIZIAN MODELS

Suppose $T$ is a completion of $ZF + LM$ which has a well-founded model. It is natural to wonder about the cardinality of the set $\Lambda(T)$ of isomorphism types of countable *well-founded* Leibnizian models of $T$. As observed by Paris [P], every model of $T$ all of whose ordinals are definable, must be well-founded. Hence, by Lemma 2.1.5, $|\Lambda(T)| > 0$ . Furthermore, by Lemmas 2.1.5 and 2.1.6, if $T$ proves $\mathbf{V} \neq \mathbf{OD}$, then $|\Lambda(T)| = 2^{\aleph_0}$. On the other hand, if $T$ proves $\mathbf{V} = \mathbf{OD}$, then $\mathfrak{M}_T$ might be the *only* well-founded model of $T$ up to isomorphism so it is possible to have $|\Lambda(T)| = 1$ (e.g., if $T$ is the theory of the Shepherdson-Cohen minimal model). More generally, it is shown in [En-4] that under reasonable conditions, for each cardinal $\kappa \in I = \omega \cup \{\aleph_0, \aleph_1, 2^{\aleph_0}\}$, there is a completion $T_\kappa$

of $ZF + \mathbf{V} = \mathbf{OD}$ with exactly $\kappa$ nonisomorphic countable well-founded models. In particular for every $\kappa \in I$, $|\Lambda(T_\kappa)| \leq \kappa$. This motivates the following question:

**Question 4.1** Is there a completion $T$ of $ZF + \mathbf{V} = \mathbf{OD}$ such that

$$1 < |\Lambda(T)| < 2^{\aleph_0} \ ?$$

More generally, what are the possible values of $|\Lambda(T)|$ as $T$ ranges over completions of $ZF + \mathbf{V} = \mathbf{OD}$?

We conclude with remarks on *uncountable* well-founded Leibnizian models. The example below shows that it is possible for a theory with an uncountable well-founded model to lack an uncountable Leibnizian well-founded model:

**Example 4.2.** If $\aleph_1 \geq \aleph_2^{\mathbf{L}} = (\aleph_2$ of the constructible universe $\mathbf{L})$ holds in the meta-theory, then the theory $ZF + \mathbf{V} = \mathbf{L}$ has no uncountable well-founded Leibnizian model. If not, by Gödel's condensation lemma, there is some $\alpha \geq \aleph_1$ such that $L_\alpha$ is Leibnizian. But since $\left(2^{\aleph_0} = \aleph_1\right)^{\mathbf{L}}$, as viewed from $\mathbf{L}$, the cardinality of $L_\alpha$ is larger than that of the continuum and therefore, by an elementary counting argument, there are indiscernibles in $L_\alpha$, contradiction.

On the other hand, by Solovay's Lemma 3.1.3, every well-founded model $\mathfrak{M}$ of $ZF + \mathbf{V} = \mathbf{OD}$ has a Leibnizian elementary submodel $\mathfrak{N}$ with the same reals as $\mathfrak{M}$ (since every $n \in \omega^{\mathfrak{M}}$ is definable in $\mathfrak{M}$). Therefore

**Theorem 4.3.** (Solovay) *If $T$ is a completion of $ZF + \mathbf{V} = \mathbf{OD}$ which has a well-founded model with $\left|\mathbb{R}^{\mathfrak{M}}\right| = \theta$, then there is a Leibnizian well-founded model of $T$ of power $\theta$.*

In light of Theorems 4.3 and 3.3, one might ask whether there are any $\aleph_1$-like well-founded Leibnizian models of $ZF$. The following result shows that the answer is in the negative.

**Theorem 4.4** *If $\mathfrak{M}$ is a well-founded Leibnizian model of $ZF$, then the cofinality of $\mathbf{Ord}^{\mathfrak{M}}$ is $\aleph_0$.*

**Proof:** Recall that within $ZF$, $\Sigma_n$-truth is $\Sigma_n$-definable for $n \geq 1$ [J, Section 14]. Therefore, by the reflection theorem, for each natural number $n$,

(1) $\quad ZF \vdash \forall \delta \in \mathbf{Ord}\ \exists \alpha \in \mathbf{Ord}\ (\delta \in \alpha \wedge V_\alpha \prec_n \mathbf{V})$.

Here "$V_\alpha \prec_n \mathbf{V}$" is a single first order sentence of set theory expressing the statement "for all $\Sigma_n$ formulae $\varphi(v_0, \cdots, v_k)$, and all $a_0, \cdots, a_k$ in $V_\alpha$, $\varphi(a_0, \cdots, a_k)$ holds iff its relativization $\varphi^{V_\alpha}(a_0, \cdots, a_k)$ to $V_\alpha$ holds". Therefore given any model $\mathfrak{M}$ of $ZF$, there is a sequence $(\alpha_n : n \in \omega)$ of ordinals of $\mathfrak{M}$ such that

(2) $\forall n \in \omega \ \alpha_n < \alpha_{n+1} \ V^{\mathfrak{M}}_{\alpha_n} \prec_n \mathfrak{M}$.

Hence

(3) $V^{\mathfrak{M}}_{\alpha_1} \prec_1 V^{\mathfrak{M}}_{\alpha_2} \prec_2 \cdots \prec_{n-1} V^{\mathfrak{M}}_{\alpha_n} \prec_n V^{\mathfrak{M}}_{\alpha_{n+1}} \prec_{n+1} \cdots$ .

Now suppose that $\mathfrak{M}$ is a well-founded Leibnizian model of $ZF$, and suppose to the contrary that the cofinality of $\mathbf{Ord}^{\mathfrak{M}}$ is uncountable. Since $\mathbf{Ord}^{\mathfrak{M}}$ is well-founded and the sequence of $\alpha_n$'s is bounded in $\mathbf{Ord}^{\mathfrak{M}}$, there is an ordinal $\delta \in \mathbf{Ord}^{\mathfrak{M}}$ such that $\delta = \sup\limits_{n\in\omega} \alpha_n$. Therefore, (3) implies that $V^{\mathfrak{M}}_{\delta} \prec \mathfrak{M}$, which in turn shows that $V^{\mathfrak{M}}_{\delta}$ is Leibnizian. This contradicts the fact that $V^{\mathfrak{M}}_{\delta}$ is not Leibnizian since

$$\mathfrak{M} \models |V_\delta| = \beth_\delta > 2^{\aleph_0}.$$

□ (Theorem 4.4)

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