ω-Models of Finite Set Theory

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SEP 5, 2008

SOME HISTORY

- In 1953, Kreisel and Mostowski independently showed that certain finitely axiomatizable systems of set theory formulated in an *expansion* of the usual language $\{\in\}$ of set theory do not possess any recursive models. This result was improved in 1958 by Rabin who found a "familiar" finitely axiomatizable first order theory that has no recursive model: Gödel-Bernays set theory GB without the axiom of infinity (note that GB can be formulated in the language $\{\in\}$ with no extra symbols).

- These discoveries overshadowed Tennenbaum's celebrated 1961 theorem that characterizes the standard model of PA (Peano arithmetic) as the only recursive model of PA up to isomorphism.

- Mancini and Zambella's 2001 paper focuses on *Tennenbaum phenomena in set theory*. Mancini and Zambella introduced a weak fragment (dubbed $\mathsf{KP}\Sigma_1$) of Kripke-Platek set theory KP, and showed that the only recursive model of $\mathsf{KP}\Sigma_1$ up to isomorphism is the standard one, i.e., $(V_\omega, \in)$, where V_ω is the set of hereditarily finite sets.

- In contrast, they used the *Bernays-Rieger* permutation method to show that the theory $\mathsf{ZF}_{\mathsf{fin}}$ obtained by replacing the axiom of infinity by its negation in the usual axiomatization of ZF (Zermelo-Fraenkel set theory) has a recursive nonstandard model.

- The Mancini-Zambella recursive nonstandard model also has the curious feature of being an ω-*model* in the sense that every element of the model, as viewed externally, has at most finitely many members.

PRELIMINARIES

Definitions/Observations:

(a) Models of set theory are *directed graphs* (hereafter: *digraphs*), i.e., structures of the form $\mathfrak{M} = (M, E)$, where E is a binary relation on M that interprets $\in$. We often write xEy as a shorthand for $\langle x, y\rangle \in E$. For $c \in M$, c_E is the set of "elements" of c, i.e.,

$$c_E := \{m \in M : mEc\}.$$

$\mathfrak{M}$ is *nonstandard* if E is not well-founded, i.e., if there is a sequence $\langle c_n : n \in \omega\rangle$ of elements of M such that $c_{n+1}Ec_n$ for all $n \in \omega$.

(b)

$$\text{EST} := \text{Ext.} + \text{Empty Set} + \text{Pairs} + \text{Union} + \text{Repl.}$$

(c) The theory $\mathsf{ZF_{fin}}$ is obtained by replacing the axiom of infinity by its negation in the usual axiomatization of ZF (Zermelo-Fraenkel set theory). More explicitly:

$$\mathsf{ZF_{fin}} := \mathsf{EST} + \mathsf{Power\ set} + \mathsf{Regularity} + \neg\mathsf{Infinity}.$$

Here Infinity is the usual axiom of infinity, i.e.,

$$\mathsf{Infinity} := \exists x \left(\emptyset \in x \wedge \forall y\,(y \in x \to y^{+} \in x)\right),$$

where $y^{+} := y \cup \{y\}$.

(d) $\mathsf{Tran}(x)$ is the first order formula that expresses the statement "x is transitive", i.e.,

$$\mathsf{Tran}(x) := \forall y \forall z\,(z \in y \in x \to z \in x).$$

(e) $\mathsf{TC}(x)$ is the first order formula that expresses the statement "the transitive closure of x is a set", i.e.,

$$\mathsf{TC}(x) := \exists y\,(x \subseteq y \wedge \mathsf{Tran}(y)).$$

Overtly, the above formula just says that some superset of x is transitive, but it is easy to see that $\mathsf{TC}(x)$ is equivalent within EST to the following statement expressing "there is a smallest transitive set that contains x"

$$\exists y\,(x \subseteq y \wedge \mathsf{Tran}(y) \wedge \forall z\,((x \subseteq z \wedge \mathsf{Tran}(z)) \to y \subseteq z)).$$

(f) TC denotes the *transitive closure* axiom

$$\mathsf{TC} := \forall x\ \mathsf{TC}(x).$$

Let V_ω be the set of hereditarily finite sets. It is easy to see that $\mathsf{ZF}_{\mathsf{fin}} + \mathsf{TC}$ holds in V_ω. However, it has "long been known" that $\mathsf{ZF}_{\mathsf{fin}} \nvdash \mathsf{TC}$.

(g) $\mathbb{N}(x)$ [read as "x is a natural number"] is the formula

$$\mathsf{Ord}(x) \wedge \forall y{\in}x^{+}\,(y \neq \emptyset \to \exists z\,(\mathsf{Ord}(z) \wedge y = z^{+})),$$

where $\mathsf{Ord}(x)$ expresses "x is a (von Neumann) ordinal", i.e., "x is a transitive set that is well-ordered by $\in$". It is well-known that with this interpretation, the full induction scheme $\mathsf{Ind}_{\mathbb{N}}$, consisting of the universal closure of formulas of the following form is provable within EST

$$\Big(\theta(0) \wedge \forall x \left(\mathbb{N}(x) \wedge \theta(x) \to \theta(x^{+})\right)\Big) \to \forall x \left(\mathbb{N}(x) \to \theta(x)\right).$$

This can be used to show that $\mathsf{ZF}_{\mathsf{fin}}$ is *essentially reflexive*, i.e., any consistent extension of $\mathsf{ZF}_{\mathsf{fin}}$ proves the consistency of each of its finite subtheories. By Gödel's second incompleteness theorem, therefore, $\mathsf{ZF}_{\mathsf{fin}}$ is not finitely axiomatizable.

(h) For a model $\mathfrak{M} \models$ EST, and $x \in M$, we say that x is $\mathbb{N}$-*finite*if there is a bijection in $\mathfrak{M}$ between x and some element of $\mathbb{N}^{\mathfrak{M}}$. Let:

$$(V_\omega)^{\mathfrak{M}} := \{m \in M : \mathfrak{M} \models \mathsf{TC}(m) \wedge x \text{ is } \mathbb{N}\text{-finite}\}$$

It is easy to see that

$$(V_\omega)^{\mathfrak{M}} \models \mathsf{ZF}_{\mathsf{fin}} + \mathsf{TC}.$$

(i) $\tau(n, x)$ is the term expressing "the n-th approximation to the transitive closure of $\{x\}$ (where n is a natural number)". Informally speaking,

$\tau(0, x) = \{x\}$;

$\tau(n + 1, x) = \tau(n, x) \cup \{y : \exists z\, (y \in z \in \tau(n, x)\}$.

Thanks to the coding apparatus of EST for dealing with finite sequences, and the provability of $\mathrm{Ind}_{\mathbb{N}}$ within EST (both mentioned earlier in part (g)), the above informal recursion can be formalized within EST to show that

$$\mathsf{EST} \vdash \forall n \forall x \, (\mathbb{N}(n) \to \exists! y \, (\tau(n,x) = y)).$$

This leads to the following important observation:

(j) Even though the transitive closure of a set need not form a set in EST (or even in $\mathsf{ZF}_{\mathsf{fin}}$), for an ω-model $\mathfrak{M}$ the transitive closure $\tau(c)$ of $\{c\}$ is *first order definable* via:

$$\tau^{\mathfrak{M}}(c) := \{m \in M : \mathfrak{M} \vDash \exists n \, (\mathbb{N}(n) \wedge m \in \tau(n,c))\}.$$

This shows that, in the worst case scenario, transitive closures behave like proper classes in ω-models of $\mathfrak{M}$.

$\mathsf{Fin}_{\mathbb{N}}$

- Let $\mathsf{Fin}_{\mathbb{N}}$:= Every set is can be put into 1-1 correspondence with a finite ordinal.

- $\mathsf{Fin}_{\mathbb{N}}$ is provable within $\mathsf{ZF}_{\mathsf{fin}} \setminus$ {Regularity} (i.e., within EST + Powerset + ¬ Infinity).

- EST + $\mathsf{Fin}_{\mathbb{N}}$ + Regularity axiomatizes the same first order theory as $\mathsf{ZF}_{\mathsf{fin}}$.

KAYE-WONG (2008)

Kaye and Wong showed that within EST, the principle TC is equivalent to the scheme of $\in$-Induction consisting of statements of the following form (θ is allowed to have suppressed parameters)

$$\forall y\,(\forall x{\in}y\ \theta\,(x) \to \theta(y)) \to \forall z\,\theta(z).$$

They also showed the following strong form of bi-interpretability between PA and $\mathsf{ZF_{fin}} + \mathsf{TC}$, known as *definitional equivalence* (or *synonymity*) by showing that:

(1) TC holds in the Ackermann interpretation Ack of $\mathsf{ZF_{fin}}$ within PA, i.e., $\mathsf{Ack} : \mathsf{ZF_{fin}} + \mathsf{TC} \to \mathsf{PA}$; and

(2) Ack is *invertible*, i.e., there is an interpretation $\mathsf{B} : \mathsf{PA} \to \mathsf{ZF_{fin}} + \mathsf{TC}$ such that $\mathsf{Ack} \circ \mathsf{B} = \mathsf{id_{PA}}$ and $\mathsf{B} \circ \mathsf{Ack} = \mathsf{id}_{\mathsf{ZF_{fin}}+\mathsf{TC}}$.

BUILDING ω-MODELS

Definition. Suppose $\mathfrak{M}$ is a model of EST. $\mathfrak{M}$ is an *ω-model* if $|x_E|$ is finite for every $x \in M$ satisfying $\mathfrak{M} \models$ "x is $\mathbb{N}$-finite".

The following proposition provides a useful *graph-theoretic* characterization of *ω-models of* $\mathsf{ZF}_{\mathsf{fin}}$. Note that even though $\mathsf{ZF}_{\mathsf{fin}}$ is not finitely axiomatizable, the equivalence of (a) and (b) of Proposition 3.2 shows that there is a single sentence in the language of set theory whose ω-models are precisely ω-models of $\mathsf{ZF}_{\mathsf{fin}}$.

- Recall that a vertex x of a digraph $G := (X, E)$ has *finite in-degree* if x_E is finite; and G is *acyclic* if there is no finite sequence $x_1 \; E \; x_2 \; \cdots \; E \; x_{n-1} \; E \; x_n$ in G with $x_1 = x_n$.

Proposition *The following three conditions are equivalent for a digraph $G := (X, E)$:*

(a) G *is an* ω*-model of* $\mathsf{ZF}_{\mathsf{fin}}$.

(b) G *is an* ω*-model of* Extensionality, Empty set, Regularity, Adjunction *and* $\neg$Infinity.

(c) G *satisfies the following four conditions:*

(i) E *is extensional;*

(ii) *Every vertex of* G *has finite in-degree*;

(iii) *G is acyclic; and*

(iv) *G has an element of in-degree 0, and for all positive* $n \in \omega$,

$$(X, E) \models \forall x_1 \cdots \forall x_n \; \exists y \; \forall z \; (zEy \leftrightarrow \bigvee_{i=1}^{n} z = x_i).$$

Definition Suppose $G := (X, E)$ is an extensional, acyclic digraph, all of whose vertices have finite in-degree.

(a) A subset S of X is said to be *coded* in G if there is some $x \in X$ such that $S = x_E$.

(b) $D(G) := \{S \subseteq X : S$ is finite and S is not coded in $G\}$. We shall refer to $D(G)$ as the *deficiency set of* G.

(c) Without loss of generality, we assume that for *all* considered digraphs $G = (X, E)$, $X \cap D(G) = \emptyset$.

(d) The infinite sequence of digraphs

$$\langle \mathbb{V}_n(G) : n \in \omega \rangle \, ,$$

where $\mathbb{V}_n(G) := (V_n(G), E_n(G))$, is built recursively using the following clauses:

$$V_0(G) := X; \quad E_0(G) := E.$$

$$V_{n+1}(G) := V_n(G) \cup D(\mathbb{V}_n(G));$$

$$E_{n+1}(G) := E_n(G) \cup$$

$$\{\langle x, X \rangle \in V_n(G) \times D(\mathbb{V}_n(G)) : x \in X\}.$$

(e) $\mathbb{V}_\omega(G) := (V_\omega(G),\ E_\omega(G))$, where

$$V_\omega(G) := \bigcup_{n \in \omega} V_n(G), \quad E_\omega(G) := \bigcup_{n \in \omega} E_n(G).$$

Theorem. *If* $G := (X, E)$ *is an extensional, acyclic digraph, all of whose vertices have finite in-degree, then* $\mathbb{V}_\omega(G)$ *is an* ω*-model of* $\mathsf{ZF}_{\mathsf{fin}}$.

Examples

(a) For every transitive $S \subseteq V_\omega$, $\mathbb{V}_\omega(S, \in) \cong (V_\omega, \in)$.

(b) Let $G_\omega := (\omega, \{\langle n+1, n\rangle : n \in \omega\})$. $\mathbb{V}_\omega(G_\omega)$ *is our first concrete example of a nonstandard* ω*-model of* $\mathsf{ZF}_{\mathsf{fin}}$.

Corollary . $\mathsf{ZF}_{\mathsf{fin}}$ *has* ω*-models in every infinite cardinality.*

In contrast with $\mathsf{ZF_{fin}} + \mathsf{TC}$, within $\mathsf{ZF_{fin}}$ there is no definable bijection between the universe and the set of natural numbers. Furthermore, since $\mathbb{V}_\omega(\{0,1\} \times G_\omega)$ has an automorphism of order 2, there is not even a definable linear ordering of the universe available in $\mathsf{ZF_{fin}}$.

We need to introduce a key definition before stating the next result:

- A digraph $G = (\omega, E)$ is said to be *highly recursive* if for each $n \in \omega$, n_E is finite, and the map $n \mapsto c(n_E)$ is recursive, where c is a canonical code for n_E. Clearly if G is highly recursive, then the edge-set E of G is recursive.

Corollary. $\mathsf{ZF_{fin}}$ *has nonstandard highly recursive ω-models.*

Remark. A minor modification of the proof of the above Corollary shows that there are infinitely many pairwise elementarily nonequivalent highly recursive models of $\mathsf{ZF}_{\mathsf{fin}}$. In particular, this shows that in contrast to PA and $\mathsf{ZF}_{\mathsf{fin}}+\mathsf{TC}$, $\mathsf{ZF}_{\mathsf{fin}}+\neg\mathsf{TC}$ has infinitely many nonisomorphic recursive models. However, as shown by the next theorem, $\mathsf{ZF}_{\mathsf{fin}}$ does not entirely escape the reach of Tennenbaum phenomena.

Theorem. *Every recursive model of* $\mathsf{ZF}_{\mathsf{fin}}$ *is an* ω*-model.*

Recall that two theories U and V are said to be *bi-interpretable* if there are interpretations $\mathcal{I} : U \to V$ and $\mathcal{J} : V \to U$, a binary U-formula F, and a binary V-formula G, such that F is, U-verifiably, an isomorphism between id_U and $\mathcal{J} \circ \mathcal{I}$, and G is, V-verifiably, an isomorphism between id_V and $\mathcal{I} \circ \mathcal{J}$. This notion is entirely syntactic, but has several model theoretic ramifications. In particular, given models $\mathfrak{A} \models U$ and $\mathfrak{B} \models V$, the interpretations $\mathcal{I}$ and $\mathcal{J}$ give rise to (1) models $\mathfrak{A}^{\mathcal{J}} \models V$ and $\mathfrak{B}^{\mathcal{I}} \models U$, and (2) isomorphisms $F^{\mathfrak{A}}$ and $G^{\mathfrak{B}}$ with

$$F^{\mathfrak{A}} : \mathfrak{A} \xrightarrow{\cong} \left(\mathfrak{A}^{\mathcal{J}}\right)^{\mathcal{I}} \text{ and } G^{\mathfrak{B}} : \mathfrak{B} \xrightarrow{\cong} \left(\mathfrak{B}^{\mathcal{I}}\right)^{\mathcal{J}}.$$

A much weaker syntactic notion, dubbed *sentential equivalence*, is obtained by replacing the demand on the existence of definable isomorphisms with the requirement that the relevant models be elementarily equivalent, i.e., for any $\mathfrak{A} \models U$ and $\mathfrak{B} \models V$,

$$\mathfrak{A} \equiv \left(\mathfrak{A}^{\mathcal{J}}\right)^{\mathcal{I}} \text{ and } \mathfrak{B} \equiv \left(\mathfrak{B}^{\mathcal{I}}\right)^{\mathcal{J}}.$$

Theorem $\mathsf{ZF}_{\mathsf{fin}}$ *and* PA *are not sententially equivalent*.

Proof: Suppose to the contrary that the interpretations

$$\mathcal{I} : \mathsf{ZF}_{\mathsf{fin}} \to \mathsf{PA}, \text{ and } \mathcal{J} : \mathsf{PA} \to \mathsf{ZF}_{\mathsf{fin}}$$

witness the sentential equivalence of $\mathsf{ZF}_{\mathsf{fin}}$ and PA. In light of the fact that there are at least two nonelementarily equivalent recursive models of $\mathsf{ZF}_{\mathsf{fin}}$, in order to reach a contradiction it is sufficient to demonstrate that the hypothesis about $\mathcal{I}$ and $\mathcal{J}$ can be used to show that any two arithmetical ω-models of $\mathsf{ZF}_{\mathsf{fin}}$ are elementarily equivalent. To this end, suppose $\mathfrak{M}$ is an arithmetical ω-model of $\mathsf{ZF}_{\mathsf{fin}}$. We claim that

$$\mathfrak{M}^{\mathcal{J}} \cong (\omega, +, \times).$$

The following classical theorem of Scott plays a key role in the verification of our claim. In what follows, an *arithmetical model of* PA refers to a structure of the form $(\omega, \oplus, \otimes)$, where there are first order formulas $\varphi(x, y, z)$ and $\psi(x, y, z)$ that respectively define the graphs of the binary operations $\oplus$ and $\otimes$ in the model $(\omega, +, \times)$.

Theorem (Scott, 1960). *No arithmetical nonstandard model of* PA *is elementarily equivalent to* $(\omega, +, \cdot)$.

Let $\mathfrak{M}' \equiv \left(\mathfrak{M}^{\mathcal{J}}\right)^{\mathcal{I}}$. By assumption, $\mathfrak{M}' \equiv \mathfrak{M}$, which implies that

$$(\mathbb{N}, +, \times)^{\mathfrak{M}} \equiv (\mathbb{N}, +, \times)^{\mathfrak{M}'}.$$

This shows that $(\mathbb{N},+,\times)^{\mathfrak{M}'}$ is elementarily equivalent to $(\omega,+,\cdot)$ since $\mathfrak{M}$ is an ω-model. Coupled with the fact that $(\mathbb{N},+,\times)^{\mathfrak{M}'}$ is an arithmetical model (since it is arithmetically interpretable in an arithmetical model), Scott's aforementioned theorem can be invoked to show that $\mathfrak{M}'$ must be an ω-model. But since no nonstandard model of PA can interpret an isomorphic copy of $(\omega,<)$ this shows that for any arithmetical ω-model $\mathfrak{M}$ of $\mathsf{ZF}_{\mathsf{fin}}$, $\mathfrak{M}^{\mathcal{J}} \cong (\omega,+,\times)$. Therefore, the assumption of sentential equivalence of $\mathsf{ZF}_{\mathsf{fin}}$ and PA implies that any two arithmetical ω-models of $\mathsf{ZF}_{\mathsf{fin}}$ are elementarily equivalent, which is the contradiction we were aiming to arrive at. $\square$

MODELS WITH SPECIAL PROPERTIES

Theorem. *For every graph* (A, F) *there is an* ω*-model* $\mathfrak{M}$ *of* $\mathsf{ZF}_{\mathsf{fin}}$ *whose universe contains* A *and which satisfies the following conditions:*

(a) (A, F) *is definable in* $\mathfrak{M}$;

(b) *Every element of* $\mathfrak{M}$ *is definable in* $(\mathfrak{M}, x)_{x \in A}$;

(c) *If* (A, F) *is pointwise definable, then so is* $\mathfrak{M}$;

(d) $\mathsf{Aut}(\mathfrak{M}) \cong \mathsf{Aut}(A, F)$.

Corollary. *Every group can be realized as the automorphism group of an ω-model of* $\mathsf{ZF}_{\mathsf{fin}}$.

Corollary. *For every infinite cardinal κ there are 2^κ nonisomorphic rigid ω-models of* $\mathsf{ZF}_{\mathsf{fin}}$ *of cardinality κ.*

Corollary. *For every infinite cardinal κ there is a family $\mathcal{M}$ of cardinality 2^κ* of *ω-models of* $\mathsf{ZF}_{\mathsf{fin}}$ *of cardinality κ such that for any distinct $\mathfrak{M}_1$ and $\mathfrak{M}_2$ in $\mathcal{M}$, there is no elementary embedding from $\mathfrak{M}_1$ into $\mathfrak{M}_2$.*

Corollary. *There are $2^{\aleph_0}$ pointwise definable ω-models of* $\mathsf{ZF}_{\mathsf{fin}}$. *Consequently there are $2^{\aleph_0}$ complete extensions of* $\mathsf{ZF}_{\mathsf{fin}}$ *that possess ω-models.*

Corollary. *For every structure* $\mathfrak{A}$ *in a finite signature there is an* ω*-model* $\mathfrak{M}$ *of* $\mathsf{ZF}_{\mathsf{fin}}$ *such that* $\mathfrak{M}$ *interprets an isomorphic copy of* $\mathfrak{A}$*, and* $\mathsf{Aut}(\mathfrak{A}) \cong \mathsf{Aut}(\mathfrak{B})$*. Furthermore, if* $\mathfrak{A}$ *is pointwise definable, then so is* $\mathfrak{M}$*.*

Corollary. *There are infinitely many countable nonisomorphic* ω*-models of* $\mathsf{ZF}_{\mathsf{fin}}$ *each of which is the unique* ω*-model of some finite extension of* $\mathsf{ZF}_{\mathsf{fin}}$*.*