A STANDARD MODEL OF PEANO ARITHMETIC WITH NO CONSERVATIVE ELEMENTARY EXTENSION

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Abstract. The principal result of this paper answers a long-standing question in the model theory of arithmetic [KS, Question 7] by showing that there exists an uncountable arithmetically closed family $\mathcal{A}$ of subsets of the set ω of natural numbers such that the expansion $\Omega_{\mathcal{A}} := (\omega, +, \cdot, X)_{X\in\mathcal{A}}$ of the standard model of Peano arithmetic has no conservative elementary extension, i.e., for any elementary extension $\Omega^*_{\mathcal{A}} = (\omega^*, \cdots)$ of $\Omega_{\mathcal{A}}$, there is a subset of ω^* that is parametrically definable in $\Omega^*_{\mathcal{A}}$ but whose intersection with ω is not a member of $\mathcal{A}$.

Inspired by a recent question of Gitman and Hamkins, we also show that the aforementioned family $\mathcal{A}$ can be arranged to further satisfy the curious property that forcing with the quotient Boolean algebra $\mathcal{A}/FIN$ (where FIN is the ideal of finite sets) collapses $\aleph_1$ when viewed as a notion of forcing.

1. INTRODUCTION

By the celebrated MacDowell-Specker theorem every model of PA (Peano arithmetic) of any cardinality has an elementary end extension. Gaifman and Phillips independently refined this result by showing that every model of PA has a *conservative* elementary end extension, in other words, every model $\mathfrak{M} = (M, \oplus^{\mathfrak{M}}, \otimes^{\mathfrak{M}})$ of PA has an elementary end extension $\mathfrak{N} = (N, \oplus^{\mathfrak{N}}, \otimes^{\mathfrak{N}})$ such that for any $X \subseteq N$ that is definable in $\mathfrak{N}$, $X \cap M$ is definable in $\mathfrak{M}$ (n.b., throughout the paper "definable" means parametrically definable). Indeed the Gaifman-Phillips result holds for any model of $PA(\mathcal{L})$, as long as $\mathcal{L}$ is a *countable* language extending the language of arithmetic[1]. Here $PA(\mathcal{L})$ is the extension of PA obtained by adding all instances of the induction scheme for $\mathcal{L}$-formulae. This prompted Gaifman to raise the following questions for uncountable languages $\mathcal{L}$.

Question 1.1. (Gaifman [Ga])

(a) *Does every model of $PA(\mathcal{L})$ have an elementary end extension?*
(b) *Does every model of $PA(\mathcal{L})$ have a conservative elementary end extension?*

In 1978 Mills [M] used a novel forcing construction to answer Question 1.1 in the negative. Starting with any countable *nonstandard* model $\mathfrak{M}$ of PA and an infinite element $a \in M$, Mills used forcing to construct an uncountable family $\mathcal{F}$ of functions from M into $\{m \in M : m < a\}$ such that (1) the expansion $(\mathfrak{M}, f)_{f\in\mathcal{F}}$

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[1] See [Ka, Theorem 8.6], or [KS, Sec. 2.2] for an exposition of the Gaifman-Phillips result.

satisfies PA in the extended language employing a name for each $f \in \mathcal{F}$, and (2) for any distinct f and g in $\mathcal{F}$, there is some $b \in M$ such that $f(x) \neq g(x)$ for all $x \geq b$. It is easy to see that (2) implies that $(\mathfrak{M}, f)_{f\in\mathcal{F}}$ has no proper elementary end extension. Since it is well-known that conservative elementary extensions of models of $PA(\mathcal{L})$ are automatically end extensions, this provides a negative answer to both parts of Question 1.1.

Blass observed that the first order theory of the model constructed by Mills has no *standard* model, an observation which leads to the following refinement of Question 1.1(b) pertaining to expansions of the standard model of arithmetic, i.e., models of the form

$$\Omega_{\mathcal{A}} := (\omega, +, \cdot, X)_{X\in\mathcal{A}},$$

where $\mathcal{A}$ is a family of subsets of ω.

Question 1.2. (Blass, [KS, Question 7])

Is there $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that $\Omega_{\mathcal{A}}$ has no conservative elementary extension ?

The principal result of this paper (Theorem A, Section 2) provides a positive answer to Question 1.2. In Section 3 we present two further theorems in the realm of the model theory of arithmetic inspired by Theorem A: Theorem B demonstrates that the assumption of countability cannot be removed from a classical theorem of Kirby and Paris concerning strong cuts, and Theorem C complements Mills' aforementioned solution to Question 1.1 by establishing the existence of an uncountable model $\mathfrak{M}$ of $PA(\mathcal{L})$ with $|M| = |\mathcal{L}| = \aleph_1$ such that $\mathfrak{M}$ has no elementary end extension (the existence of such a model was anticipated by Mills [M, Sec. 3] but our construction is quite different). In Section 4 we discuss the curious relationship between Theorem A and a recent question in set theory posed by Gitman and Hamkins dealing with proper notions of forcing. Finally, in Section 5 we present and discuss three open problems.

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2. THE MAIN RESULT

In this section we shall establish the following theorem.

Theorem A. *There is a family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ of cardinality $\aleph_1$ such that $\Omega_{\mathcal{A}}$ has no conservative elementary extension.*

The proof of Theorem A relies on a number of different results and techniques in set theory and model theory. Let us first take at look at the high-level summary of the key ideas of the proof.

Idea (1). By a theorem of Pincus and Solovay [PS, Theorem 2], assuming $\mathrm{Con}(ZF)$, there is a model of $ZF + DC$ (dependent choice) in which $\mathcal{P}(\omega)$ does not carry a nonprincipal ultrafilter[2]. Moreover, the Pincus-Solovay proof can be used to show that if there is an ω-model of ZF (i.e., a model with no nonstandard integers), then there is a countable ω-model $\mathfrak{M}_0$ of $ZF{+}DC$ that satisfies the statement "there is no nonprincipal ultrafilter on $\mathcal{P}(\omega)$". Therefore, if $\mathcal{A}_0 := (\mathcal{P}(\omega))^{\mathfrak{M}_0}$ and $\mathbb{N} := (\omega, +, \cdot)$, then $(\mathbb{N}, \mathcal{A}_0)$ is an ω-model of second order arithmetic Z_2 plus the full scheme of dependent choice (Π^1_∞-DC) that satisfies the key property: no nonprincipal ultrafilter $\mathcal{U}$ over the Boolean algebra $\mathcal{A}_0$ is definable within $(\mathbb{N}, \mathcal{A}_0)$ [3].

Idea (2). Suppose $\mathcal{A} \subseteq \mathcal{P}(\omega)$ and $(\mathbb{N}, \mathcal{A}) \models ACA_0$ (i.e., $\mathcal{A}$ is closed under arithmetical definability). Let us say that a family $\mathcal{S} \subseteq \mathcal{P}(\omega)$ is *piecewise coded in* $\mathcal{A}$ if for every $X \in \mathcal{A}$ there is some $Y \in \mathcal{A}$ such that

$$\{n \in \omega : (X)_n \in \mathcal{S}\} = Y,$$

where $(X)_n$ is the n-th real coded by the real X. We shall use an omitting types argument[4] employing Jensen's combinatorial principle $\diamondsuit_{\aleph_1}$ to show that any countable ω-model $(\mathbb{N}, \mathcal{A})$ of Z_2 plus the schema Π^1_∞-AC (this is not a typo, AC suffices here) has an elementary extension $(\mathbb{N}, \mathcal{B})$ such that the only piecewise coded subsets $\mathcal{S}$ of $\mathcal{B}$ are those that are definable in $(\mathbb{N}, \mathcal{B})$. The proof of this result takes advantage of a canonical correspondence between models of $Z_2 + \Pi^1_\infty$-AC, and models of ZFC^- + "all sets are finite or countable" (here ZFC^- is ZFC without the power set axiom).

Idea (3). The key connection between ideas (1) and (2) is provided by the following equivalence for an arithmetically closed family $\mathcal{A} \subseteq \mathcal{P}(\omega)$: $\Omega_{\mathcal{A}}$ has a conservative elementary extension iff there is a nonprincipal ultrafilter $\mathcal{U} \subseteq \mathcal{A}$ such that $\mathcal{U}$ is piecewise coded in $\mathcal{A}$. Thus, starting with the family $\mathcal{A}_0$ of idea (1), idea (2) can be used to build an elementary extension $(\mathbb{N}, \mathcal{B})$ of $(\mathbb{N}, \mathcal{A}_0)$ in order to establish Theorem A in $ZFC + \diamondsuit_{\aleph_1}$.

Idea (4). By implementing a trick borrowed from Schmerl [Sch-1], an absoluteness theorem of Shelah [Sh-1] can be invoked in order to establish Theorem A within ZFC alone.

This concludes the summary of the main ideas of the proof of Theorem A, and we are now ready to flesh out the above outline. Let us begin with a review of some preliminaries pertaining to second order arithmetic.

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[2]Indeed, in the Pincus-Solovay model, no set carries a nonprincipal ultrafilter. One can use other models of set theory as well: recall that by a classical theorem of Sierpinski, a nonprincipal ultrafilter on $\mathcal{P}(\mathbb{N})$ - viewed as a subset of the Cantor set - is neither Lebesgue measurable nor has the Baire property. Therefore, there is no nonpricipal ultrafilter on $\mathcal{P}(\mathbb{N})$ in Solovay's celebrated model [So] of $ZF + DC$ + "all sets of reals Lebesgue measurable". One can also use Shelah's model [Sh-2] of $ZF + DC$ in which all sets of reals have the Baire property (the construction of Solovay's model in [So] requires the consistency of ZF plus "there is an inaccessible cardinal", but Shelah's model only requires Con(ZF)).

[3]For logical purists: with more work, one can force directly over models of second order arithmetic and reduce the assumption of the existence of an ω-model of ZF in this step to the existence of an ω-model of Z_2.

[4]Our omitting types argument is inspired by a construction of Rubin and Shelah [RS], and generalizes a theorem due independently to Mostowski and Keisler [Ke-2, Chapter 28] stating that every countable ω-model of Z_2 plus Π^1_∞-AC has an elementary ω-extension of power $\aleph_1$.

- The systems Z_2 and ACA_0 are as in Simpson's encyclopedic reference [Si]. Z_2 is often referred to as *second order arithmetic*[5], or as *analysis*. ACA_0 is the subsystem of Z_2 with the comprehension scheme limited to formulae with no second order quantifiers.
- Models of second order arithmetic (and its subsystems) are of the *two-sorted* form $(\mathfrak{M}, \mathcal{A})$, where $\mathfrak{M}$ is a model of a fragment of PA and $\mathcal{A}$ is a family of subsets of M. Note that if $(\mathfrak{M}, \mathcal{A}) \models ACA_0$, then $(\mathfrak{M}, S)_{S\in\mathcal{A}} \models PA(\mathcal{L})$.
- The Choice Scheme Π^1_∞-AC consists of the universal closure of formulae of the form $\forall n\ \exists X\ \varphi(n, X) \to \exists Y\ \forall n\ \varphi(n, (Y)_n)$ where $\varphi(n, X)$ is a formula of second order arithmetic in which Y does not occur free.

Our first lemma is folklore and provides a key translation of "$\Omega_\mathcal{A}$ has a conservative elementary extension" in terms of the existence of piecewise coded[6] ultrafilters over $\mathcal{A}$. See [E, Lemma 3.5] for a proof.

Lemma 2.1. *The following two statements are equivalent for* $(\mathbb{N}, \mathcal{A}) \models ACA_0$:
(a) $\Omega_\mathcal{A}$ *has a conservative elementary extension.*
(b) *There is a nonprincipal ultrafilter* $\mathcal{U} \subseteq \mathcal{A}$ *such that* $\mathcal{U}$ *is piecewise coded in* $\mathcal{A}$.

Guided by Lemma 2.1, we now work towards the construction of a model $(\mathbb{N}, \mathcal{A})$ of ACA_0 such that no nonprincipal ultrafilter $\mathcal{U} \subseteq \mathcal{A}$ is piecewise coded in $\mathcal{A}$. The following lemma lies at the heart of our construction. In order to state it, we first need a general model theoretic definition:

- Suppose $\mathfrak{M} = (M, \cdots)$. Two disjoint subsets V and W of M are *inseparable* in $\mathfrak{M}$ if there is no $D \subseteq M$ such that D is definable in $\mathfrak{M}$, $V \subseteq D$, and $W \cap D = \varnothing$.

Lemma 2.2. *Suppose* $(\mathbb{N}, \mathcal{A})$ *is a countable* ω*-model of* $Z_2 + \Pi^1_\infty$-AC*, and*

$$\{\{\mathcal{V}_n, \mathcal{W}_n\} : n \in \omega\}$$

is a countable list of pairs of inseparable subsets of $\mathcal{A}$. *There is an elementary extension* $(\mathbb{N}, \mathcal{B})$ *of* $(\mathbb{N}, \mathcal{A})$ *which satisfies the following two properties:*

(a) $\mathcal{V}_n$ *and* $\mathcal{W}_n$ *remain inseparable in* $(\mathbb{N}, \mathcal{B})$ *for all* $n \in \omega$.
(b) *There is some* $X \in \mathcal{B}$ *such that* $\mathcal{A} \subseteq \{(X)_n : n \in \omega\}$, *where* $(X)_n$ *is the* n*-th real coded by the real* X.

In order to establish Lemma 2.2 we first make an important conceptual change of venue, and move from models of second order arithmetic to the equivalent context of models of an appropriate set theory. This transition allows us to concentrate on the key ideas of the proof without having to worry about routine but laborious coding arguments. What allows us to safely make this transition is the well-known canonical one-to-one correspondence[7] (explained in detail in [Si, VII.3]) between models of the two theories

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[5]Some authors, especially those belonging to the Polish school of logic, use A_2^- for the system Z_2 (and A_2 for Z_2 plus the choice scheme).

[6]Piecewise coded ultrafilters are dubbed *iterable* in [E] (since ultrapowers based on them are amenable to iteration along any linear order). To make matters more confusing, the same ultrafilters are referred to as *definable* in [Ki] since they correspond to definable types.

[7]This correspondence was first explicitly noted by Mostowski in the context of the so-called β-models of $T_{analysis}$ (which correspond to well-founded models of T_{set}). Simpson [Si, VII.3] has refined the correspondence between models of $T_{analysis}$ and T_{set} by identifying set-theoretic equivalents of various subsystems of $T_{analysis}$ that contain ATR_0.

$$T_{analysis} := Z_2 + \Pi^1_\infty\text{-}AC, \text{ and } T_{set} := ZFC^- + \mathbf{V} = H(\aleph_1)$$

(here $\mathbf{V} = H(\aleph_1)$ is the set theoretic assertion "all sets are finite or countable"). More specifically, in order to canonically interpret a model $\mathfrak{A} \models T_{set}$ within a model $(\mathfrak{M}, \mathcal{A}) \models T_{analysis}$, one first defines the notion of "suitable trees" [Si, Def. VII.3.10], and then one defines an equivalence relation $=^*$ among suitable trees, and a binary relation $\in^*$ among the equivalence classes of $=^*$ [Si, Def. VII.3.13] in order to obtain a model $\mathfrak{A} = (A, E)$ of T_{set} (where A is the set of equivalence classes of $=^*$ and $E = \in^*$). Moreover, a routine calculation shows that for each $X \subseteq \mathcal{A}_0$, the above process produces a corresponding subset $\widehat{X}$ of A_0 such that X is definable in $(\mathbb{N}, \mathcal{A}_0)$ iff $\widehat{X}$ is definable in $\mathfrak{A}_0$. Conversely, if $\mathfrak{A}_0 \models T_{set}$, then the standard model of second order arithmetic in the sense of $\mathfrak{A}$ is a model of $T_{analysis}$, and for any $Y \subseteq A_0$ there is a corresponding $\widetilde{Y} \subseteq \mathcal{A}_0$ such that Y is definable in $\mathfrak{A}_0$ iff $\widetilde{Y}$ is definable in $(\mathbb{N}, \mathcal{A}_0)$. It is also easy to see that ω-models of $T_{analysis}$ correspond to ω-models of T_{set}. The "synonymity" between $T_{analysis}$ and T_{set} allows us to reformulate Lemma 2.2 as follows:

Lemma 2.2. (set theoretic formulation). *Suppose $\mathfrak{A} = (A, E)$ is a countable ω-model of T_{set}, and $\{\{V_n, W_n\} : n \in \omega\}$ is a countable list of inseparable pairs of subsets of A. There exists an elementary extension $\mathfrak{B} = (B, F)$ of $\mathfrak{A}$ that satisfies the following three properties:*

(a) *V_n and W_n remain inseparable in $\mathfrak{B}$ for all $n \in \omega$.*
(b) *There is some $c \in B$, such that $A \subseteq c_F := \{b \in B : \mathfrak{B} \models b \in c\}$.*
(c) *$\mathfrak{B}$ is an ω-model (i.e., $\mathfrak{B}$ has no nonstandard integers).*

Let $\mathcal{L}$ be the language $\{\in\}$ augmented with constants $\{c\} \cup \{\overline{a} : a \in A\}$, and let

$$T := Th(\mathfrak{A}, a)_{a \in A} + \{a \subseteq c : a \in A\}.$$

Of course T is consistent since it is finitely satisfiable in $\mathfrak{A}$ (this only uses the axioms of Pairs and Sumset to invoke the closure of $\mathfrak{A}$ under finite unions). Moreover, if $\mathfrak{B} \models T$, then $\mathfrak{A} \prec \mathfrak{B}$ and $\mathfrak{B}$ satisfies condition (b) of the theorem since T proves $a \in c$ for each $a \in A$ because T proves $a \in \{a\} \subseteq c$. To arrange a model of T in which conditions (a) and (c) also hold requires a delicate omitting types argument. First, we need a pair of preliminary lemmas:

Lemma 2.2.1. *The following two conditions are equivalent for a sentence $\varphi(c)$ of $\mathcal{L}$.*

(i) *$T \vdash \varphi(c)$.*
(ii) *$\mathfrak{A} \models \exists r \forall s (r \subseteq s \rightarrow \varphi(s))$.*

Proof: Left to the reader.
□
Lemma 2.2.1 immediately yields:

Lemma 2.2.2. *The following two conditions are equivalent for a sentence $\varphi(c)$ of $\mathcal{L}$.*

(i) *$T + \varphi(c)$ is consistent.*
(ii) *$\mathfrak{A} \models \forall r \exists s (r \subseteq s \wedge \varphi(s))$.*

□

We are now ready to carry out our omitting types arguments. Consider the following set of 1-types formulated in the language $\mathcal{L}$:

- $\Gamma(x) = \{\text{“}x \in \omega\text{”}\} \cup \{x \neq n : n \in \omega\}$. Here “$x \in \omega$” stands for the usual formula in the language of set theory expressing “x is a finite von Neumann ordinal”.
- For each formula $\psi(t, x)$ of $\mathcal{L}$, and each $n \in \omega$,

$$\Sigma_n^{\psi}(x) := \{\psi(\overline{v}, x) : v \in V_n\} \cup \{\neg\psi(\overline{w}, x) : w \in W_n\}.$$

Note that Σ_n^{ψ} expresses:

$$\text{“}V_n \subseteq \{t : \psi(t, x)\} \text{ and } W_n \subseteq \{t : \neg\psi(t, x)\}\text{”}.$$

Lemma 2.2.3. $\Gamma(x)$ *is locally omitted by* T.

Proof: As noted earlier, T proves $a \in c$ for each $a \in A$ since T proves $a \in \{a\} \subseteq c$. Therefore Lemma 2.2.3 follows from Lemma 2.2.2 and the fact that the replacement schema holds in $\mathfrak{A}$, precisely as in [CK, Theorem 2.2.18].
□

Lemma 2.2.4. Σ_n^{ψ} *is locally omitted by* T *for each formula* $\psi(t, x)$*, and each* $n \in \omega$.

Proof: Suppose that, on the contrary, there is a formula $\theta(x, c)$ of $\mathcal{L}$ and some $n \in \omega$ such that (1) - (3) below hold:

(1) $T + \exists x\theta(x, c)$ is consistent.

(2) For all $v \in V_n$, $T \vdash \theta(x, c) \to \psi(\overline{v}, x)$.

(3) For all $w \in W_n$, $T \vdash \theta(x, c) \to \psi(\overline{w}, x)$.

Invoking Lemmas 2.2.1 and 2.2.2, (1) through (3) translate to the following (note the introduction of formulae $\lambda(.)$ and $\gamma(.)$):

(1′) $\mathfrak{A} \models \forall r \exists s(r \subseteq s \wedge \exists x\theta(x, s))$.

(2′) For all $v \in V_n$, $\mathfrak{A} \models \overbrace{\exists r \forall s\, (r \subseteq s \to (\theta(x, s) \to \psi(\overline{v}, x)))}^{\lambda(\overline{v})}$.

(3′) For all $w \in W_n, \mathfrak{A} \models \overbrace{\exists r \forall s\, (t \subseteq x \to (\theta(x, s) \to \neg\psi(\overline{w}, x)))}^{\gamma(\overline{w})}$.

Let $\Lambda := \{a \in A : \mathfrak{A} \models \lambda(\overline{a})\}$, and $\Gamma := \{a \in A : \mathfrak{A} \models \gamma(\overline{a})\}$, and observe that $V_n \subseteq \Lambda$ by (2′) and $W_n \subseteq \Gamma$ by (3′). We aim to establish that $\Lambda \cap \Gamma = \emptyset$, which implies that V_n and W_n are separable in $\mathfrak{A}$, thus concluding the proof of Lemma 2.2.4. To this end, suppose to the contrary that for some $a \in A$,

(4) $\mathfrak{A} \models \lambda(\overline{a}) \wedge \gamma(\overline{a})$.

It is easy to see, using the fact that $\mathfrak{A}$ is closed under finite unions, that (4) implies:

(5) $\mathfrak{A} \models \exists r \forall s\, (r \subseteq s \to (\theta(x, s) \to (\psi(\overline{a}, x) \wedge \neg\psi(\overline{a}, x))))$.

This completes the proof since (1′) and (5) are contradictory.
□

Proof of Lemma 2.2 (set theoretic formulation): Putting Lemma 2.2.3, Lemma 2.2.4, and the Henkin-Orey omitting types theorem [CK, Theorem 2.2.9] together, there exists a model $\mathfrak{B}$ of T that satisfies properties (a) and (c). This completes the proof since as noted earlier, every model of T satisfies condition (b).

□

In order to state the next result we need to review some definitions.

- Suppose model $\mathfrak{A} = (A, E)$ is a model of some brand of set theory. A subset X of A is *coded* in $\mathfrak{A}$ if for some $a \in A$,
$$X = a_E := \{x \in A : \mathfrak{A} \models x \in a\}.$$
- $S \subseteq A$ is said to be a *class* of $\mathfrak{A} = (A, E)$ if for any $a \in A$, there is some $b \in A$ such that
$$S \cap a_E = b_E.$$
It is easy to see that if $\mathfrak{A}$ satisfies the separation scheme of set theory, then every definable subset of A is a class of $\mathfrak{A}$.
- $\mathfrak{A}$ is said to be *rather classless* if every class of $\mathfrak{A}$ is definable in $\mathfrak{A}$.

We are now in a position to state and prove the following central theorem.

Theorem 2.3.

(a) *Every countable ω-model $(\mathbb{N}, \mathcal{A}_0)$ of $T_{analysis}$ has an elementary extension $(\mathbb{N}, \mathcal{A})$ of cardinality $\aleph_1$ such that the piecewise coded subsets of $\mathcal{A}$ are precisely the subsets of $\mathcal{A}$ that are definable in $(\mathbb{N}, \mathcal{A})$.*

(b) *Every countable ω-model $\mathfrak{A}_0$ of T_{set} has a rather classless elementary extension of cardinality $\aleph_1$ which is also an ω-model.*

Proof: The discussion preceding the set theoretic formulation of Lemma 2.2 can be used to show that parts (a) and (b) of Theorem 2.3 are equivalent[8], once we point out the following additional features of the correspondence between models of $T_{analysis}$ and T_{set} :

(1) X is piecewise coded in $\mathcal{A}_0$ iff $\widehat{X}$ is a class of $\mathfrak{A}_0$, and
(2) Y is a class of $\mathfrak{A}_0$ iff $\widetilde{Y}$ is piecewise coded in $\mathcal{A}_0$.

Therefore we shall only establish part (b). The proof of part (b) has two distinct stages: in the first stage we establish (b) assuming $\diamondsuit_{\aleph_1}$, and then in the second stage we eliminate $\diamondsuit_{\aleph_1}$ with an absoluteness argument.

Stage 1: Fix a $\diamondsuit_{\aleph_1}$ sequence $\langle S_\alpha : \alpha < \omega_1 \rangle$. Given a countable ω-model $\mathfrak{A}_0$ of T_{set}, assume without loss of generality that $A_0 = \alpha_0 \in \omega_1$. We plan to inductively build two sequences $\langle \mathfrak{A}_\alpha : \alpha < \omega_1 \rangle$, and $\langle \mathcal{O}_\alpha : \alpha < \omega_1 \rangle$. The first is a sequence of approximations to our final model $\mathfrak{A}$. The second sequence, on the other hand, keeps track of the increasing list of "obligations" we need to abide by throughout the construction of the first sequence. More specifically, each $\mathcal{O}_\alpha$ will be of the form $\{\{V_n^\alpha,\ W_n^\alpha\} : n \in \omega\}$, where $\{V_n^\alpha,\ W_n^\alpha\}$ is pair of disjoint subsets A_α that are inseparable in $\mathfrak{A}_\alpha$ and should be kept inseparable in each $\mathfrak{A}_\beta$, for $\beta \geq \alpha$. We shall only describe the construction of these two sequences for stages $\alpha + 1$ for α limit since:

- $\mathcal{O}_0 := \emptyset$.
- For limit α, $\mathfrak{A}_\alpha := \bigcup\limits_{\beta<\alpha} \mathfrak{A}_\beta$ and $\mathcal{O}_\alpha := \bigcup\limits_{\beta<\alpha} \mathcal{O}_\beta$.
- For nonlimit α , $\mathfrak{A}_{\alpha+1} := \mathfrak{A}_\alpha$ and $\mathcal{O}_{\alpha+1} := \mathcal{O}_\alpha$.

[8]Note that the proof of Theorem A only relies on the implication $(b) \Rightarrow (a)$ of Theorem 2.3.

At stage $\alpha+1$, where α is a limit ordinal, we have access to a model $\mathfrak{A}_\alpha$ (where $A_\alpha \in \omega_1$), and a collection $\mathcal{O}_\alpha$ of inseparable pairs of subsets of A_α. We now look at S_α, and consider two cases: either S_α is parametrically undefinable in $\mathfrak{A}_\alpha$, or not. In the latter case we "do nothing" and define $\mathfrak{A}_{\alpha+1} := \mathfrak{A}_\alpha$ and $\mathcal{O}_{\alpha+1} := \mathcal{O}_\alpha$. But if the former is true, we augment our list of obligations via:

$$\mathcal{O}_{\alpha+1} := \mathcal{O}_\alpha \cup \{\{S_\alpha,\ A_\alpha \backslash S_\alpha\}\}.$$

Notice that if S_α is parametrically undefinable in $\mathfrak{A}_\alpha$, then $\{S_\alpha,\ A_\alpha \backslash S_\alpha\}$ is inseparable in $\mathfrak{A}_\alpha$. Then we use Lemma 2.2 to build an elementary extension $\mathfrak{A}_{\alpha+1}$ of $\mathfrak{A}_\alpha$ such that:

(1) For each $\{V,\ W\} \in \mathcal{O}_{\alpha+1}$, V and W are inseparable in $\mathfrak{A}_{\alpha+1}$.

(2) $A_\alpha \subseteq \{x \in A_{\alpha+1} : \mathfrak{A}_{\alpha+1} \models x \in c\}$ for some $c \in A_{\alpha+1}$.

(3) $\mathfrak{A}_{\alpha+1}$ is an ω-model, and $A_{\alpha+1} = A_\alpha + \omega$ (ordinal addition).

This concludes the description of the sequences $\langle \mathfrak{A}_\alpha : \alpha < \omega_1 \rangle$, and $\langle \mathcal{O}_\alpha : \alpha < \omega_1 \rangle$. Let $\mathfrak{A} := \bigcup\limits_{\alpha<\omega_1} \mathfrak{A}_\alpha$, $\mathcal{O} := \bigcup\limits_{\alpha<\omega_1} \mathcal{O}_\alpha$, and notice that

(4) For each $\{V,\ W\} \in \mathcal{O}$, V and W are inseparable in $\mathfrak{A}$.

We now verify that $\mathfrak{A}$ is rather classless. Suppose, on the contrary, that $S \subseteq A$ is an undefinable class of $\mathfrak{A}$. By usual Löwenheim-Skolem arguments there is some limit $\alpha < \omega_1$ such that

$$S \cap \alpha = S_\alpha \text{ and } (\mathfrak{A}_\alpha, S_\alpha) \prec (\mathfrak{A}, S).$$

In particular,

(5) S_α is an undefinable subset of $\mathfrak{A}_\alpha$.

Moreover, based on (2) for some $c \in A_{\alpha+1}$, $S_\alpha \subseteq \{x \in A : \mathfrak{A} \models x \in c\}$. Since S_α is assumed to be a class of $\mathfrak{A}$, there is some $d \in A$ such that

(6) $S_\alpha \cap \{x \in A : \mathfrak{A} \models x \in c\} = \{x \in A : \mathfrak{A} \models x \in d\}$.

We have arrived at a contradiction since on one hand, based on (2) and (6), the formula $\varphi(x) := x \in d$ witnesses the separability of S_α and $A_\alpha \backslash S_\alpha$ within $\mathfrak{A}$, and on the other hand $\{S_\alpha,\ A_\alpha \backslash S_\alpha\} \in \mathcal{O}$ by (5), and therefore (4) dictates that S_α and $A_\alpha \backslash S_\alpha$ are inseparable in $\mathfrak{A}$. This contradiction shows that $\mathfrak{A}$ has no undefinable classes, as desired.

Stage 2: The proof of Theorem 2.3 in ZFC relies on coupling the proof presented in stage 1 with a remarkable absoluteness theorem of Shelah. Before stating Shelah's theorem, let us review the following definitions.

- A *ranked tree* τ is a two sorted structure $\tau = (T,\ \leq_T,\ L,\ \leq_L,\ \rho)$ satisfying the following three properties:
  (1) $(T,\ \leq_T)$ is a tree, i.e., a partial order such that any two elements below a given element are comparable;
  (2) $(L,\ \leq_L)$ is a linear order; and
  (3) ρ is an order preserving map from $(T,\ \leq_T)$ onto $(L,\ \leq_L)$ with the property that for each $t \in T$, ρ maps the set of predecessors of t *onto* the initial segment determined by $\rho(t)$.

- A linearly ordered subset B of T is said to be a *branch* of τ if the image of B under ρ is L. The *cofinality* of τ is the cofinality of $(L, \leq_L)$.
- Given a structure $\mathfrak{A}$ in a language $\mathcal{L}$, and a ranked tree τ, we write $\tau = \mathbf{t}^{\mathfrak{A}}$ if $\mathbf{t}$ is an appropriate sequence of $\mathcal{L}$-formulae whose components define the corresponding components of τ in $\mathfrak{A}$.

Theorem (Shelah's absoluteness theorem [Sh-1, Theorem 6]) *Suppose $\mathcal{L}$ is a countable language, and $\mathbf{t}$ is a sequence of $\mathcal{L}$-formulae that defines a ranked tree in some $\mathcal{L}$-model. Given any sentence ψ of $\mathcal{L}_{\omega_1,\omega}(Q)$, where Q is the quantifier "there exists uncountably many", there is a countable expansion $\overline{\mathcal{L}}$ of $\mathcal{L}$, and a sentence $\overline{\psi} \in \overline{\mathcal{L}}_{\omega_1,\omega}(Q)$ such that the following two conditions are equivalent:*

(1) *$\overline{\psi}$ has a model.*
(2) *ψ has a model $\mathfrak{A}$ of power $\aleph_1$ which has the property that $\mathbf{t}^{\mathfrak{A}}$ is a ranked tree of cofinality $\aleph_1$ and every branch of $\mathbf{t}^{\mathfrak{A}}$ is definable in $\mathfrak{A}$.*

Consequently, by Keisler's completeness theorem for $\mathcal{L}^*_{\omega_1,\omega}(Q)$ [Ke-1], *(2) is an absolute statement.*

As observed in [Ke-3, Example 2.1], if $\mathfrak{A}$ is a model of ZF, then there is a definable ranked tree $\mathbf{t}^{\mathfrak{A}}$ of $\mathfrak{A}$ such that there is a canonical correspondence between the branches of $\mathbf{t}^{\mathfrak{A}}$ and the classes of $\mathfrak{A}$. The construction of $\mathbf{t}^{\mathfrak{A}}$ relies on the power set axiom since it based on the von Neumann V_α hierarchy, therefore such a canonical correspondence need not exist for arbitrary models of ZFC^-. This suggests at first sight that Shelah's absoluteness theorem is powerless in eliminating $\diamondsuit_{\aleph_1}$. However, the fact that the model $\mathfrak{A}$ produced in the first stage of the proof of Theorem 2.3 contains a *cofinal* sequence $\langle c_\alpha : \alpha < \omega_1\rangle$ of elements (in the sense that the c_α's are linearly ordered by $\in$ (and also by containment), and for each $a \in A$ there is some α such that $\mathfrak{A} \models a \in c_\alpha$) can be taken advantage of in order to bring Shelah's absoluteness theorem to bear on the situation at hand. To see this, let $\overline{\mathfrak{A}}$ be the expansion[9] of $\mathfrak{A}$ that codes up $\langle c_\alpha : \alpha < \omega_1\rangle$, i.e.,

$$\overline{\mathfrak{A}} := (\mathfrak{A},\ C,\ \lhd),$$

where $C := \{c_\alpha : \alpha < \omega_1\}$ and $\lhd$ is the ordering of C defined by: $c_\alpha \lhd c_\beta$ iff $\alpha < \beta$. Let $\overline{\mathcal{L}}$ be the language appropriate to the model $\overline{\mathfrak{A}}$, and consider the ranked tree

$$\mathbf{t}_{\mathbf{0}}^{\overline{\mathfrak{A}}} = (T_0,\ \leq_{T_0},\ L_0,\ \leq_{L_0},\ \rho_0),$$

where T_0 consists of functions f in A mapping some c_α into $\{0,1\}$, $L_0 := \{c_\alpha : \alpha < \omega_1\}$, $\leq_T$ is defined by set inclusion, $\leq_{L_0}$ is defined by set membership, and for any $f \in T_0$, $\rho_0(f)$ is the domain of f. It is easy to see that:

(♣) If $\overline{\mathfrak{B}} \equiv \overline{\mathfrak{A}}$ and $\mathfrak{B}$ is the $\in$-reduct of $\overline{\mathfrak{B}}$, then $\mathfrak{B}$ is rather classless iff every branch of $\mathbf{t}_{\mathbf{0}}^{\overline{\mathfrak{B}}}$ is definable in $\overline{\mathfrak{B}}$.

Recall that $\diamondsuit_{\aleph_1}$ holds in inner models of the form $\mathbf{L}(r)$ [Ku, Exercise 7, Ch. VI], where r is a real. Since any countable model (in a countable language) can be coded by a real, there is some real r_0 such that $\mathbf{L}(r_0)$ is a model of ZFC containing $\mathfrak{A}_0$ in which $\diamondsuit_{\aleph_1}$ holds[10]. Therefore, by the proof in Stage 1, $\mathbf{L}(r_0)$ believes that there is an ω-model $\mathfrak{A}$ that is a rather classless elementary extension of $\mathfrak{A}_0$, and $\mathfrak{A}$

[9] I am grateful to Schmerl for reminding me of this expansion trick, first introduced in [Sch-1].
[10] Alternatively, one can use forcing to arrange $\diamondsuit_{\aleph_1}$ [Ku, Theorem 8.3].

has an expansion $\overline{\mathfrak{A}}$ as above in $\mathbf{L}(r_0)$. It is easy to see that the salient features of $\overline{\mathfrak{A}}$ are expressible in $\overline{\mathcal{L}}_{\omega_1,\omega}(Q)$, i.e., there is a sentence ψ of $\overline{\mathcal{L}}_{\omega_1,\omega}(Q)$ that expresses the conjunction of the following statements (i) through (iii) about $\overline{\mathfrak{A}}$:

(i) $\mathfrak{A}$ is an elementary extension of $\mathfrak{A}_0$;
(ii) $\mathfrak{A}$ is an ω-model; and
(iii) $(C, \lhd)$ is $\aleph_1$-like, and for every $a \in A$ there is some $c \in C$ such that $\mathfrak{A} \models a \in c$.

We can now invoke Shelah's absoluteness theorem to conclude that there is a real-world model $\overline{\mathfrak{B}}$ of ψ with the property that all the branches of the ranked tree $\tau^{\overline{\mathfrak{B}}}$ are definable in $\overline{\mathfrak{B}}$, and the $\in$-reduct $\mathfrak{B}$ of $\overline{\mathfrak{B}}$ is an ω-model that is an elementary extension of $\mathfrak{A}_0$. Since $\mathfrak{B}$ is rather classless by (♣), this completes the proof of Theorem A (in ZFC alone).
□

Based on the above results, we can now present a succinct proof of Theorem A:

Proof of Theorem A. Let $(\mathbb{N}, \mathcal{A}_0)$ be a countable model of $Z_2 + \Pi^1_\infty$-AC such that no nonprincipal ultrafilter on $\mathcal{A}$ is definable in $(\mathbb{N}, \mathcal{A})$. Use Theorem 2.3 to construct an elementary extension $(\mathbb{N}, \mathcal{A})$ of $(\mathbb{N}, \mathcal{A}_0)$ with the property that no nonprincipal ultrafilter over $\mathcal{A}$ is piecewise coded in $\mathcal{A}$. Therefore by Theorem 2.1, $\Omega_{\mathcal{A}}$ has no conservative elementary extension.

Remark 2.4.

(a) The proof of Theorem A shows that Theorem A can be strengthened to show that the family $\mathcal{A}$ can be arranged to extend any prescribed countable collection of subsets of ω.

(b) The proof of Theorem 2.3 (b) does not depend on the inclusion of the axiom $\mathbf{V} = H(\aleph_1)$ in T_{set} and therefore Theorem 2.3(b) remains valid upon replacing T_{set} by ZFC^-. Indeed, Theorem 2.3(b) can be further strengthened by replacing ZFC^- by ZFC^{--} in which the replacement scheme is weakened to the scheme asserting that definable image of any set is *contained* in some set.

(c) Since every expansion of $\mathbb{N}$ has an elementary end extension, Theorem A shows that for uncountable $\mathcal{L}$, it is possible to have a countable model of $PA(\mathcal{L})$ that has an elementary end extension, but lacks a conservative elementary extension. With more work (and using the strengthening of Theorem 2.3(b) mentioned in the second sentence of the (b) above, it is also possible to use the results of this section to build a *nonstandard* model of $PA(\mathcal{L})$ which has an elementary end extension, but not a conservative elementary extension (this answers a question of Schmerl).
□

3. TWO FURTHER COUNTEREXAMPLES

In this section we present two results (Theorems B and C) that were inspired by Theorem A and which highlight the role of countability in two classical theorems in the model theory of arithmetic. In order to situate Theorem B we need to review some preliminary definitions and results. In what follows, suppose $\mathfrak{M}$ is a model of PA.

- Let $E(x, y)$ be the formula in the language of arithmetic that expresses "the x-th digit in the binary expansion of y is 1". A subset X of M is *coded* if for some $c \in M$,

$$X = c_E := \{x \in M : xEc\}.$$

It is well-known that a subset X of M is coded iff X is bounded and definable within $\mathfrak{M}$.

- I is a *cut* of $\mathfrak{M}$ if I is an initial segment of $\mathfrak{M}$ with no last element. A cut I is *strong in* $\mathfrak{M}$ if for each function f whose graph is coded in $\mathfrak{M}$ and whose domain includes I, there is some s in M such that for all $m \in M$, $f(m) \notin I$ iff $s < f(m)$.
- If I is a proper cut of $\mathfrak{M}$, then

$$SSy_I(\mathfrak{M}) := \{c_E \cap I : c \in M\}.$$

When $I = \omega$, we shall write $SSy(\mathfrak{M})$ instead of $SSy_\omega(\mathfrak{M})$. It is easy to see that $SSy(\mathfrak{M})$ is always a *Scott family*, i.e., $(\mathbb{N}, SSy(\mathfrak{M})) \models WKL_0$, where WKL_0 is the well-known subsystem of second order arithmetic [Si].

The following theorem was established by Scott [Sco] for $|\mathcal{A}| = \aleph_0$, and by Nadel-Knight [KN] and others[11] for $|\mathcal{A}| = \aleph_1$. Its status for $|\mathcal{A}| > \aleph_1$ is a major open problem (assuming that $2^{\aleph_0} > \aleph_1$).

Theorem 3.1 *If $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is a Scott family with $|\mathcal{A}| \leq \aleph_1$, then $\mathcal{A} = SSy(\mathfrak{M})$ for some model $\mathfrak{M}$ of PA.*

The other preliminary result we need is the following result of Kirby and Paris[12].

Theorem 3.2. (Kirby-Paris) *The following are equivalent for a cut I of $\mathfrak{M} \models PA$:*
(a) *I is strong in $\mathfrak{M}$.*
(b) $(\mathbf{I}, SSy_I(\mathfrak{M})) \models ACA_0$.

The point of departure for the first result of this section (Theorem B) is the following theorem. In what follows, $\mathfrak{N}$ is an *I-extension of* $\mathfrak{M}$, if $\mathfrak{N} \succ \mathfrak{M}$, I is a proper cut of $\mathfrak{M}$, and the following three conditions hold:

(1) if $x \in N$, and $x < y \in I$, then $x \in I$.
(2) There is an element c of N such that $I < c < M \setminus I$.
(3) $SSy_I(\mathfrak{M}) = SSy_I(\mathfrak{N})$.

Theorem 3.3. (Kirby-Paris [KS, Theorem 6.3.5]) *If I is a strong cut of a countable model $\mathfrak{M}$ of PA, then $\mathfrak{M}$ has an I-extension $\mathfrak{N}$.*

It is known that the converse of Theorem 3.3 holds for all models $\mathfrak{M}$ of PA irrespective of the cardinality of $\mathfrak{M}$. We now show that the countability assumption cannot be dropped from Theorem 3.3, even when $I = \omega$.

Theorem B. *There is a model $\mathfrak{M}$ of PA of cardinality $\aleph_1$ such that ω is strong in $\mathfrak{M}$, but $\mathfrak{M}$ does not have an ω-extension.*

---

[11]According to Smorynski [Sm, Theorem 2.11], the $|\mathcal{A}| = \aleph_1$ case of Theorem 3.1 was independently established by Guaspari.

[12]The original proof in [KP, Proposition 8] only establishes that a strong cut is a model of PA, but the strategy of the proof can be used to establish the stronger result, see [E, Lemma A.4] for more detail.

Proof: Let $\mathcal{A}$ be the family of size $\aleph_1$ constructed in Theorem A and recall that there is no nonprincipal ultrafilter $\mathcal{U} \subseteq \mathcal{A}$ such that $\mathcal{U}$ is piecewise coded in $\mathcal{A}$. Since $\mathcal{A}$ is arithmetically closed, $\mathcal{A}$ is a Scott set and therefore by Theorem 3.1 there is a model $\mathfrak{M}$ of PA such that $SSy(\mathfrak{M}) = \mathcal{A}$. Note that since $(\mathbb{N}, \mathcal{A}) \models ACA_0$, ω is a strong cut of $\mathfrak{M}$ by Theorem 3.2.

To see that $\mathfrak{M}$ does not have an ω-extension, suppose to the contrary that $\mathfrak{N}$ is an ω-extension of $\mathfrak{M}$, let $c \in N$ with $\omega < c < M \setminus \omega$, and fix some $d \in M \setminus \omega$. For any $X \in \mathcal{A}$ and $k \in M$ let us say that X is *ω-coded* by X if

$$k_E \cap \omega = X.$$

The key observation is that if X is ω-coded by both k and k', and x is any element of N such that $\omega < x < M \setminus \omega$, then

$$\mathfrak{N} \models xEk \text{ iff } \mathfrak{N} \models xEk'$$

(this follows from the assumption that $\mathfrak{M} \prec \mathfrak{N}$ and that any point of disagreement of k_E and k'_E in $\mathfrak{M}$ is nonstandard). Therefore the following definition yields a nonprincipal ultrafilter $\mathcal{U}$ on $\mathcal{A}$:

$$\mathcal{U} := \{X \in \mathcal{A} : \mathfrak{N} \models cEe \text{ for some } e \in M \text{ that } \omega\text{-codes } X\}.$$

The assumption $SSy(\mathfrak{M}) = SSy(\mathfrak{N})$ can now be invoked to verify that $\mathcal{U}$ is piecewise coded in $\mathcal{A}$. This contradicts our choice of $\mathcal{A}$ and concludes the proof.
□

Remark 3.4. Schmerl has pointed out that the proof of Theorem B can be modified to show that Theorem B can be refined in two ways. Firstly, $\mathfrak{M}$ can be arranged to be a model of any prescribed consistent extension of PA. Secondly, $\mathfrak{M}$ can be further required to be recursively saturated. The first refinement takes advantage of Remark 2.4 (a); the second is based on a variant of Theorem 3.1 in which $\mathfrak{M}$ is required to be recursively saturated (cf. [Sm, Theorem 5.12] and the parenthetical comments following [Sm, Corollary 5.14]).

The second result of this section (Theorem C) provides an example of a model of $PA(\mathcal{L})$ with no elementary end extension that is quite different from Mills' example described in the introduction. The proof of Theorem C employs a variation on the following result.

Theorem 3.5. (Rubin [RS] with $\diamondsuit_{\aleph_1}$, Schmerl [Sch-1] without $\diamondsuit_{\aleph_1}$) *Every countable model $\mathfrak{A}_0$ of ZFC has an elementary extension $\mathfrak{A}$ of power $\aleph_1$ such that every $\omega^{\mathfrak{A}}$-complete ultrafilter over $\mathfrak{A}$ is coded in $\mathfrak{A}$.*

Schmerl [Sch-1] defines $\mathcal{U}$ to be an *ultrafilter over* $\mathfrak{A}$ if $\mathfrak{A} = (A, E)$ and there is some infinite cardinal κ of $\mathfrak{A}$ such that $\mathcal{U}$ is an ultrafilter over the Boolean algebra $\{x_E : \mathfrak{A} \models x \subseteq \kappa\}$. Such a $\mathcal{U}$ is said to be *$\omega^{\mathfrak{A}}$-complete* if $\mathcal{U}$ meets all partitions of κ_E that are finite in the sense of $\mathfrak{A}$. Theorem 3.4 does not overtly address the behavior of ultrafilters over models $\mathfrak{A}$ of ZFC^-, but an analysis of its proof reveals that it can be used verbatim to establish Theorem 3.6 below. Note that the proof of Theorems 3.5 yields a model $\mathfrak{A}$ whose set of natural numbers has cofinality $\aleph_1$ and is therefore nonstandard.

Theorem 3.6. *Every countable model $\mathfrak{A}_0$ of ZFC^- has an elementary extension $\mathfrak{A}$ of power $\aleph_1$ such that every $\omega^{\mathfrak{A}}$-complete over $\mathfrak{A}$ is definable in $\mathfrak{A}$.*

We now use Theorem 3.5 the establish the following result.

Theorem C. *There is a model* $\mathfrak{M} \vDash PA(\mathcal{L})$ *with* $|M| = |\mathcal{L}| = \aleph_1$ *such that* $\mathfrak{M}$ *has no elementary end extension.*

Proof: Recall from the discussion of the Solovay-Pincus theorem in Section 2 that if ZF has an ω-model, then there is a countable $\mathcal{A}_0 \subseteq \mathcal{P}(\omega)$ such that $(\mathbb{N}, \mathcal{A}_0)$ is a model of $Z_2 + \Pi^1_\infty$-AC such that no nonprincipal ultrafilter on $\mathcal{A}_0$ is definable in $(\mathbb{N}, \mathcal{A})$. Let $\mathfrak{A}_0$ be the model of T_{set} that is canonically associated with $(\mathbb{N}, \mathcal{A}_0)$. By Theorem 3.5 there is an elementary extension $\mathfrak{A}$ of $\mathfrak{A}_0$ of power $\aleph_1$ such that every $\omega^{\mathfrak{A}}$-complete $\mathfrak{A}$-ultrafilter is definable in $\mathfrak{A}$. Therefore, by the choice of $\mathcal{A}_0$, this means that there is no nonprincipal $\omega^{\mathfrak{A}}$-complete $\mathfrak{A}$-ultrafilter.

Let $(\mathbb{N}^*, \mathcal{A}^*)$ be the canonical model of $T_{analysis}$ associated with $\mathfrak{A}$ (i.e., $(\mathbb{N}^*, \mathcal{A}^*) = (\mathbb{N}, \mathcal{P}(\omega))^{\mathfrak{A}}$). The desired model of $PA(\mathcal{L})$ with no elementary end extension is

$$(\mathbb{N}^*,\ X)_{X \in \mathcal{A}^*}.$$

This is easy to see since if

$$(\mathbb{N}^*,\ X)_{X \in \mathcal{A}^*} \prec_e (\overline{\mathbb{N}},\ \overline{X})_{X \in \mathcal{A}^*},$$

then one could choose $c \in \overline{\mathbb{N}} \setminus \mathbb{N}^*$ and arrive at a contradiction by producing a nonprincipal $\omega^{\mathfrak{A}}$-complete $\mathcal{U}$ ultrafilter via:

$$\mathcal{U} := \{X \in \mathcal{A}^* : c \in \overline{X}\}.$$

□

4. A QUESTION OF GITMAN AND HAMKINS

In this section we discuss the relationship between a recent question in set theory involving proper[13] posets and Theorem A. Given a Boolean algebra $\mathcal{A} \subseteq \mathcal{P}(\omega)$, let $\mathbb{P}_{\mathcal{A}}$ denote the quotient Boolean algebra $\mathcal{A}/FIN$, where FIN is the ideal of finite subsets of ω. By a classical theorem of Hausdorff, $\mathbb{P}_{\mathcal{P}(\omega)}$ is $\aleph_1$-closed and therefore $\mathbb{P}_{\mathcal{P}(\omega)}$ is a proper poset. Moreover, if $\mathcal{A}$ is countable, then $\mathbb{P}_{\mathcal{A}}$ trivially satisfies the c.c.c. condition and is therefore a proper poset (indeed, Hamkins observed that $\mathbb{P}_{\mathcal{A}}$ satisfies the c.c.c. condition iff $\mathcal{A}$ is countable, assuming that $\mathcal{A}$ satisfies some mild closure conditions much weaker than closure under arithmetically definability). Recently, Gitman ([Gi-1], [Gi-2]) used the proper forcing axiom (PFA) to show that if $\mathcal{A}$ is arithmetically closed and $\mathbb{P}_{\mathcal{A}}$ is proper, then $\mathcal{A}$ is the standard system of some model of PA. Gitman and Hamkins have conjectured that $\mathbb{P}_{\mathcal{A}}$ is a proper poset for a wide class of families $\mathcal{A}$, and asked the following question:

Question 4.1. (Gitman-Hamkins) *Is there an arithmetically closed* $\mathcal{A}$ *for which* $\mathbb{P}_{\mathcal{A}}$ *fails to be a proper poset?*

Since a proper poset preserves $\aleph_1$ when viewed as a notion of forcing, the following theorem provides a positive answer to Question 4.1.

Theorem D. *There is an uncountable arithmetically closed* $\mathcal{A} \subseteq \mathcal{P}(\omega)$ *with the property that forcing with* $\mathbb{P}_{\mathcal{A}}$ *collapses* $\aleph_1$.

---

[13]Proper posets not only preserve $\aleph_1$, but also the notion of "stationarity". They include the classes of c.c.c. and $\aleph_1$-closed posets, see [J-2] for a quick introduction.

The proof of Theorem D is based on Theorem 4.2 below, which is a refinement of Theorem 2.3. Intuitively speaking, the relationship between Theorem 2.3 and Theorem 4.2 is the same as the relationship between the theorems "there is an $\aleph_1$-Aronszajn tree" and "there is a special[14] $\aleph_1$-Aronszajn tree", since if τ_1 is an $\aleph_1$-Aronszajn tree, τ_2 is a special $\aleph_1$-Aronszajn tree, and $\mathbb{P}$ is a partial order with the property that forcing with $\mathbb{P}$ preserves $\aleph_1$, then τ_1 need not remain $\aleph_1$-Aronszajn in $\mathbf{V}^{\mathbb{P}}$ (e.g., if τ_1 is a Suslin tree and $\mathbb{P}$ is the poset obtained by reversing the order on τ_1), but τ_2 remains an $\aleph_1$-Aronszajn tree in $\mathbf{V}^{\mathbb{P}}$.

Theorem 4.2.
(a) *Every countable ω-model $(\mathbb{N}, \mathcal{A}_0)$ of $T_{analysis}$ has an elementary extension $(\mathbb{N}, \mathcal{A})$ of cardinality $\aleph_1$ such that for any partial order $\mathbb{P}$ with the property that forcing with $\mathbb{P}$ does not collapse $\aleph_1$, $\mathbf{V}^{\mathbb{P}}$ satisfies the statement "the piecewise coded subsets of $\mathcal{A}$ are precisely the subsets of $\mathcal{A}$ that are definable in* $(\mathbb{N}, \mathcal{A})$".

(b) *Every countable ω-model $\mathfrak{A}_0$ of T_{set} has a rather classless elementary extension $\mathfrak{A}$ of cardinality $\aleph_1$ such that $\mathfrak{A}$ is an ω-model and has the property that for any partial order $\mathbb{P}$ with the property that forcing with $\mathbb{P}$ does not collapse $\aleph_1$, $\mathfrak{A}$ is rather classless in* $\mathbf{V}^{\mathbb{P}}$.

Proof: Similar to Theorem 2.3, parts (a) and (b) of Theorem 4.2 are equivalent and we shall therefore only verify the veracity of part (b). The proof of part (b) is really an elaboration of the proof of part (b) of Theorem 2.3 and is obtained by analyzing a key step of the proof of Shelah's absoluteness theorem. Suppose $\mathfrak{M}$ is a structure of power $\aleph_1$ in a countable language $\mathcal{L}$ and $\tau := (T,\ \leq_T,\ L,\ \leq_L,\ \rho)$ is a definable ranked tree of $\mathfrak{M}$ whose cofinality is $\aleph_1$ and all of whose branches are definable in $\mathfrak{M}$. Generalizing the work of Baumgartner-Malitz-Reinhardt [BMR], Shelah [Sh-1] showed that there is an expansion

$$\mathfrak{M}^* = (\mathfrak{M},\ P,\ \leq_P, f)$$

of $\mathfrak{M}$ in a c.c.c. generic extension of the universe that satisfies the following three conditions:

(1) $(P,\ \leq_P)$ is isomorphic to the ordered set of rationals $\mathbb{Q}$ and $f : T \to P$;
(2) if $x <_T y$, then $f(x) \leq_P f(y)$; and
(3) if $x <_T y$ and $f(x) = f(y)$, then $\{z \in T : z \leq_T x$, or $x <_T z$ and $f(z) = f(x)\}$ is a branch of τ.

- We shall use the expression "f is a generalized specializing function for τ" to abbreviate the conjunction of (1) - (3) above.

Note that τ satisfies the following key property $(*)$:

> $(*)$ If $\mathbb{P}$ is a poset that preserves $\aleph_1$ when viewed as a notion of forcing, then $\mathbf{V}^{\mathbb{P}}$ satisfies the statement "all branches of τ are definable in $\mathfrak{M}$".

To see that $(*)$ is true, suppose $\mathbb{P}$ preserves $\aleph_1$ and B is a branch of τ in $\mathbf{V}^{\mathbb{P}}$. Then by the assumptions regarding the uncountable cofinality of τ (in $\mathbf{V}$) and the preservation of $\aleph_1$ in the passage from $\mathbf{V}$ to $\mathbf{V}^{\mathbb{P}}$, the cofinality of τ in $\mathbf{V}^{\mathbb{P}}$ is also uncountable and therefore by condition (2) f is eventually constant on B. Choose

[14]Recall: an $\aleph_1$-Aronszajn tree $(T, \leq_T)$ is *special* if there is some f mapping T into the set of rational numbers $\mathbb{Q}$ such that $x <_T y$ implies $f(x) < f(y)$.

x and y in B with $x <_T y$ and $f(x) = f(y)$. Then by condition (3), B is already in $\mathbf{V}$ and therefore definable in $\mathfrak{M}$.

In light of the above discussion, if $\overline{\mathfrak{B}}$ is the model constructed in the second stage of the proof of Theorem 2.3, there is a c.c.c. generic extension of the universe in which $\overline{\mathfrak{B}}$ has an expansion $\left(\overline{\mathfrak{B}}\right)^* = (\overline{\mathfrak{B}},\ P,\ \leq_P, f)$ such that f is a generalized specializing function for $\tau = \mathbf{t}_{\mathbf{0}}^{\overline{\mathfrak{B}}}$. Let U be a subset of L_0 of order type ω_1 that is $\leq_{L_0}$-cofinal and let $\mathcal{L}'$ be the language appropriate to the model $(\left(\overline{\mathfrak{B}}\right)^*, U)$. We can now write an $\mathcal{L}'_{\omega_1,\omega}(Q)$ sentence φ that describes the features of $\left(\overline{\mathfrak{B}}\right)^*$ that we are interested in, i.e., φ expresses the conjunction of the following sentences (1) - (5) below:

(1) $\mathfrak{A}_0 \prec \mathfrak{B}$;
(2) $\mathfrak{B}$ is an ω-model;
(3) f is a generalized specializing function for $\tau_0^{\overline{\mathfrak{B}}}$;
(4) $(U, \leq_L)$ is $\aleph_1$-like and U is $\leq_{L_0}$-cofinal in L_0;
(5) All the branches of $\tau_0^{\overline{\mathfrak{B}}}$ are definable in $\overline{\mathfrak{B}}$ (by condition (3), this can be expressed as an $\mathcal{L}'_{\omega_1,\omega}(Q)$ sentence).

By Keisler's completeness theorem for $\mathcal{L}'_{\omega_1,\omega}(Q)$, φ has a model of the form $\left(\overline{\mathfrak{A}}\right)^*$ in the real world. Therefore all the branches of $\tau_0^{\overline{\mathfrak{A}}}$ are definable in $\overline{\mathfrak{A}}$ and $\tau_0^{\overline{\mathfrak{A}}}$ is equipped with a generalized specializing function, which in light of $(*)$ shows that the $\in$-reduct $\mathfrak{A}$ of $\left(\overline{\mathfrak{A}}\right)^*$ is the desired model of part (b) of Theorem 4.2.
□

Armed with Theorem 4.2, and using the same line of argument deriving Theorem A from Theorem 2.3, we obtain the following strengthening of Theorem A:

Theorem A*. *There is a family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ of cardinality $\aleph_1$ with the property that for any partial order $\mathbb{P}$ such that forcing with $\mathbb{P}$ does not collapse $\aleph_1$, there is no nonprincipal ultrafilter in $\mathbf{V}^{\mathbb{P}}$ on $\mathcal{A}$ that is piecewise coded in $\mathcal{A}$ (consequently $\Omega_{\mathcal{A}}$ has no conservative elementary extension in $\mathbf{V}^{\mathbb{P}}$).*

We shall now derive Theorem D from Theorem A*.

Proof of Theorem D: We begin with a key definition. For subsets X and Y of ω, let us say that X *decides* Y iff either $X \subseteq_* Y$ or $X \subseteq_* \mathbb{N}\backslash Y$, where $X \subseteq_* Y$ denotes the statement "$X\backslash Y$ is finite". For any $\mathcal{A}_0 \subseteq \mathcal{P}(\omega)$ and any $X \in \mathcal{A}_0$, let

$$\mathcal{D}_X := \{[Y] \in \mathbb{P}_{\mathcal{A}_0} : \forall n \in \omega\ (Y \text{ decides } (X)_n)\}.$$

(here $[Y] = \{X \in \mathcal{A}_0 : Y \ \Delta\ X \text{ is finite}\}$, where Δ denotes symmetric difference). It is known that if $\mathcal{A}_0$ is arithmetically closed, then $\mathcal{D}_X$ is dense in $\mathbb{P}_{\mathcal{A}}$ [E, Theorem 3.4 (b)]. Now suppose $\mathbb{P}_{\mathcal{A}_0}$ is used as a notion of forcing, $\mathcal{G}$ is a generic filter, and $\mathcal{U} = \cup\mathcal{G}$ is the ultrafilter on $\mathcal{A}_0$ generated by $\mathcal{G}$, i.e., $\mathcal{U}$ consists of all elements $X \in \mathcal{A}_0$ such that $[X] \in \mathcal{G}$. It is routine to verify, using the fact that $\mathcal{G}$ meets $\mathcal{D}_X$ for every $X \in \mathcal{A}_0$, that $\mathcal{U}$ is piecewise coded in $\mathcal{A}_0$. This shows that forcing with $\mathbb{P}_{\mathcal{A}_0}$ produces a nonprincipal ultrafilter in $\mathbf{V}^{\mathbb{P}_{\mathcal{A}_0}}$ on $\mathcal{A}_0$ that is piecewise coded in $\mathcal{A}_0$. Therefore, if $\mathcal{A}_0$ is chosen as the family $\mathcal{A}$ of Theorem A*, forcing with $\mathbb{P}_{\mathcal{A}}$ collapses $\aleph_1$.
□

5. OPEN QUESTIONS

By the Gaifman-Phillips result mentioned in the introduction, if $\mathcal{A}$ is countable, then every model of $Th(\Omega_{\mathcal{A}})$ has an elementary end extension. Moreover, by a theorem of Blass [B] every model of $Th(\Omega_{\mathcal{P}(\omega)})$ (known as *full arithmetic*) has an elementary end extension. These facts motivate the following question.

Question I. *Is there $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that some model of $Th(\Omega_{\mathcal{A}})$ has no elementary end extension?*

The author conjectures that Question I can be answered in the positive by choosing $\mathcal{A}$ to be the family of Theorem A. The next question is inspired by the highly nonconstructive nature of the proofs of theorem A and D. Note that both parts of Question II have a positive answer when $\mathcal{A}$ is countable or when $\mathcal{A} = \mathcal{P}(\omega)$.

Question II. *Suppose $\mathcal{A} \subseteq \mathcal{P}(\omega)$ and $\mathcal{A}$ is Borel (where $\mathcal{A}$ is identified - via characteristic functions - as a subset of the Cantor space 2^ω).*
(a) *Does $\Omega_{\mathcal{A}}$ have a conservative elementary extension?*
(b) *Suppose, furthermore, that $\mathcal{A}$ is arithmetically closed. Is $\mathbb{P}_{\mathcal{A}}$ a proper poset?*

Two comments are in order in connection with Question II:

- As a corollary of a deep theorem of Schmerl [KS, Theorem 5.4.3], if $\mathcal{A}$ is a family of mutually Cohen generic reals over the standard model $\mathbb{N}$ of arithmetic, then $\Omega_{\mathcal{A}}$ has a conservative elementary extension. In light of the folklore fact that there is a perfect subtree τ of $2^{<\omega}$ such that any two distinct branches of τ are mutually Cohen generic over $\mathbb{N}$, this provides a nontrivial example of an uncountable *closed* $\mathcal{A} \subseteq \mathcal{P}(\omega)$ for which part (a) of Question II has a positive answer.
- Let μ be the coin-tossing measure on Borel subsets of the Cantor space 2^ω. By a classical theorem of Steinhaus, if $\mathcal{A}$ is Borel and is closed under symmetric differences, then either $\mathcal{A} = \mathcal{P}(\omega)$ or $\mu(\mathcal{A}) = 0$. Therefore in Question II one may further stipulate that $\mu(\mathcal{A}) = 0$.

In order to state and motivate our last question, we need to recall some preliminary definitions and results. Suppose $\mathcal{U}$ is an ultrafilter on $\mathcal{A} \subseteq \mathcal{P}(\omega)$ with $n \in \omega$, $n \geq 1$.

- $\mathcal{U}$ is $(\mathcal{A}, n)$-*Ramsey,* if for every $f : [\omega]^n \to \{0,1\}$ whose graph is coded in $\mathcal{A}$, there is some $X \in \mathcal{U}$ such that $f \restriction [X]^n$ is constant.
- $\mathcal{U}$ is $\mathcal{A}$-*Ramsey* if $\mathcal{U}$ is $(\mathcal{A}, n)$-*Ramsey* for all nonzero $n \in \omega$.
- $\mathcal{U}$ is $\mathcal{A}$-*minimal* iff for every $f : \omega \to \omega$ whose graph is coded in $\mathcal{A}$, there is some $X \in \mathcal{U}$ such that $f \restriction X$ is either constant or injective.

Note that every ultrafilter over any Boolean algebra $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is $(\mathcal{A}, 2)$-Ramsey. The next theorem summarizes some key relationships amongst the above notions.

Theorem 5.1[15]**.** *Suppose $\mathcal{U}$ is an ultrafilter on an arithmetically closed $\mathcal{A} \subseteq \mathcal{P}(\omega)$.*
(a) *If $\mathcal{U}$ is $(\mathcal{A}, 2)$-Ramsey, then $\mathcal{U}$ is piecewise coded in $\mathcal{A}$.*
(b) *If $\mathcal{U}$ is both piecewise coded in $\mathcal{A}$ and $\mathcal{A}$-minimal, then $\mathcal{U}$ is $\mathcal{A}$-Ramsey.*

[15]Part (a) is due to Kirby [Ki, Theorem D]; part (b) is based on the observation that Kunen's proof of "minimal implies Ramsey" for ultrapowers over $\mathcal{P}(\omega)$ can be adapted to the present context; part (c) is an immediate consequence of parts (a) and (b); part (d) is due to W. Rudin (consistency) [J-1, p.478] and Kunen (independence) [J-1, Lemma 38.1]).

(c) *If $\mathcal{U}$ is $(\mathcal{A}, 2)$-Ramsey, then $\mathcal{U}$ is $\mathcal{A}$-Ramsey.*
(d) *For $\mathcal{A} = \mathcal{P}(\omega)$, the existence of an $\mathcal{A}$-minimal ultrafilter is both consistent and independent of ZFC.*

It is also well-known that if $\mathcal{A}$ is arithmetically closed and countable, then there is a nonprincipal ultrafilter $\mathcal{U}$ on $\mathcal{A}$ that is piecewise coded in $\mathcal{A}$, $\mathcal{A}$-Ramsey, and $\mathcal{A}$-minimal. Putting part (c) of Theorem 5.1 with Theorem A we obtain:

Corollary 5.2. *There exists an arithmetically closed $\mathcal{A} \subseteq \mathcal{P}(\omega)$ of power $\aleph_1$ such that $\mathcal{A}$ carries no nonprincipal $(\mathcal{A}, 2)$-Ramsey ultrafilter.*

Corollary 5.2 and part (b) of Theorem 5.1 provide the context for our last question:

Question III. *Can it be proved in ZFC that there exists an arithmetically closed $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that $\mathcal{A}$ carries no $\mathcal{A}$-minimal ultrafilter?*

References

[BMR] J. Baumgartner, J. Malitz, and W. Reinhardt, *Embedding trees in the rationals,* **Proceedings of the National Academy of Sciences the United States of America**, vol. 67 (1970), pp. 1748–1753.

[B] A. Blass, *Conservative extensions of models of arithmetic*, **Arch. Math. Logik Grundlag.** vol. 20 (1980), no. 3-4, pp. 85–94.

[CK] C. C. Chang and H. J. Keisler, **Model Theory**, North-Holland, Amsterdam 1973.

[E] A. Enayat, *From bounded arithmetic to second order arithmetic via automorphisms*, **Logic in Tehran,** Proceedings of the Conference and Workshop of Logic, Arithmetic, and Algebra (A. Enayat. I. Kalantari, and M. Moniri, ed.), Lecture Notes in Logic Series, vol. 26, ASL and A.K. Peters, 2006.

[Ga] H. Gaifman, *Models and types of Peano's arithmetic*, **Annals of Mathematical Logic** vol. 9 (1976), pp. 223–306.

[Gi-1] V. Gitman, *Can the proper forcing axiom provide a partial solution to the Scott set problem?*, abstract of talk at the Winter meeting of the Association for Symbolic Logic (December 28, 2005, New York), **Bulletin of Symbolic Logic**, vol. 12 (2006), p. 508.

[Gi-2] *Can the proper forcing axiom give new solutions to the Scott set problem? (* manuscript), to appear.

[J-1] T. Jech, **Set Theory**, Academic Press, New York, 1978.

[J-2] ——, **Multiple Forcing,** Cambridge University Press, Oxford, 1986.

[Ka] R. Kaye, **Models of Arithmetic**, Oxford Logic Guides, vol. 15, Oxford University Press, 1991.

[Ke-1] H.J. Keisler, *Logic with the quantifier "there exist uncountably many"*, **Annals of Mathematical. Logic** vol. 1 (1970) pp. 1–93.

[Ke-2] ——, **Model Theory for Infinitary Logic,** North-Holland, Amsterdam (1971).

[Ke-3] ——, *Models with tree structures*, **Proceedings of the Tarski Symposium** (Proc. Sympos. Pure Math., Vol. XXV, Univ. California, Berkeley, Calif., 1971), pp. 331–348. Amer. Math. Soc., Providence, R.I., 1974.

[Ki] L. Kirby, *Ultrafilters and types on models of arithmetic*, **Annals of Pure and Applied Logic**, vol. 27 (1984). pp. 215-252.

[KP] L. Kirby and J. Paris, *Initial segments of models of Peano's axioms*, in **Lecture Notes in Mathematics**, vol. 619, Springer-Verlag, 1977, pp. 211-226.

[KN] J. Knight and M. Nadel, *Expansions of models and Turing degrees*, **Journal of Symbolic Logic** vol. 47 (1982), pp. 833–840.

[KS] R. Kossak and J. Schmerl, **The Structure of Models of Peano Arithmetic**, Oxford University Press, 2006.

[Ku] K. Kunen, **Set Theory**, North-Holland, Amsterdam, 1980.

[M] G. Mills, *A model of Peano arithmetic with no elementary end extension*, **Journal of Symbolic Logic** vol. 43 (1978), no. 3, pp. 563–567.

[PS] D. Pincus and R. Solovay, *Definability of measures and ultrafilters,* **Journal of Symbolic Logic,** vol. 42 (1977), pp. 179-190.

[RS] M. Rubin and S. Shelah, *On the elementary equivalence of automorphism groups of Boolean algebras; downward Skolem-Löwenheim theorems and compactness of related quantifiers*, **Journal of Symbolic Logic** vol. 45 (1980), no. 2, pp. 265–283.

[Sch-1] J. Schmerl, *Elementary extensions of models of set theory*, **Archive for Mathematical Logic** vol. 39 (2000), no. 7, pp. 509–514.

[Sco] D. Scott, *Algebras of sets binumerable in complete extensions of arithemetic*, **Recursive Function Theory** (Proc. Sympos. Pure Math., Vol. V(ed. J. C. E. Dekker), pp. 117-122, Amer. Math. Soc., Providence, R.I., 1962.

[Sh-1] S. Shelah, *Models with second order properties II. Trees with no undefined branches*, **Annals of Mathematical Logic,** vol. 14 (1978), pp. 73-87.

[Sh-2] ____, *Can you take Solovay's inaccessible away?* **Israel Journal of Mathematics**, vol. 48, pp.1-47 (1984).

[Sh-3] ____, **Proper and Improper Forcing** (second edition), Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1998.

[Si] S. Simpson, **Subsystems of second order arithmetic**, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1999.

[Sm] C. Smorynski, C. Smorynski, *Lectures on nonstandard models of arithmetic,* In **Logic Colloquium '82**, G. Lolli and A. Marcja (ed.), North Holland, Amsterdam, 1984.

[So] R. Solovay, *A model of set theory in which every set of reals is Lebesgue measurable*, **Annals of Mathematics,** vol. 92, pp. 1- 56 (1970).

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