IN PRAISE OF NONSTANDARD MODELS

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OVERVIEW

- The study of nonstandard models of set theory arise in the following contexts:

  (a) Foundations of nonstandard analysis;

  (b) Generalized quantifiers;

  (c) Consistency/independence results;

  (d) Model theory of set theory.

BASICS (1)

- Models of set theory are of the form $\mathfrak{M} = (M, E)$, where $E = \in^{\mathfrak{M}}$.

- $\mathfrak{M}$ is *standard* if E is well-founded.

- $\mathfrak{M}$ is ω*-standard* if $(\omega, <)^{\mathfrak{M}} \cong (\omega, <)$.

- **Proposition.** *$\mathfrak{M}$ is nonstandard iff*

  $(\mathbf{Ord}, \in)^{\mathfrak{M}}$ *is not well-founded.*

- **Proposition.** *Every $\mathfrak{M}$ has an elementary extension that is not ω-standard.*

BASICS (2)

- For $\mathfrak{M} = (M, E)$, and $m \in M$,

$$m_E := \{x \in M : xEm\}.$$

- Suppose $\mathfrak{M} \subseteq \mathfrak{N} = (N, F)$ with $m \in M$. $\mathfrak{N}$ is said to *fix* m if $m_E = m_F$, else $\mathfrak{N}$ *enlarges* m.

- $\mathfrak{N}$ *end extends* $\mathfrak{M}$ if $m_E = m_F$ for every $m \in M$.

- $\mathfrak{N}$ *rank extends* $\mathfrak{M}$ if for every $x \in N \backslash M$, and every $y \in M$, $\mathfrak{N} \models \rho(x) > \rho(y)$.

- **Proposition.** *Rank extensions are end extensions, but not vice-versa.*

- **Proposition.** *Elementary end extensions are rank extensions.*

Keisler-Morely Theorem

- **Theorem** [Keisler-Morley, 1968]. *Suppose* $\mathfrak{M}$ *is a countable model of: ZFC for (a) and ZC for (b).*

  (a) *For every prescribed linear order* $\mathbb{L}$, $\mathfrak{M}$ *has an elementary end extension* $\mathfrak{N}$ *which has a copy of* $\mathbb{L}$ *in* $\mathbf{Ord}^{\mathfrak{N}}$;

  (b) *If* $\kappa \in \mathbf{Ord}^{\mathfrak{M}}$ *is a prescribed regular cardinal in the sense of* $\mathfrak{M}$, *then there is an elementary extension* $\mathfrak{N} = (N, F)$ *such that* $\mathfrak{N}$ *enlarges* κ *and contains a copy of* $\mathbb{Q}$, *but* $\mathfrak{N}$ *fixes every member of* κ.

- **Corollary.** *If* $\mathfrak{M}$ *is a countable model of Z, and* $\kappa \in \mathbf{Ord}^{\mathfrak{M}}$ *is a prescribed regular cardinal in the sense of* $\mathfrak{M}$, *then there is an elementary extension* $\mathfrak{N} = (N, F)$ *such that* $\mathfrak{N}$ *enlarges* κ *and is* $\aleph_1$*-like.*

Proof of Part (b) of Keisler-Morley's Theorem

- Let $\mathbb{B}$ be the Boolean algebra $\mathcal{P}(\kappa)^{\mathfrak{M}}$ and let $\mathcal{U}$ be an ultrafilter on $\mathbb{B}$. We wish to define the (limited) ultrapower

$$\mathfrak{M}^*_{\mathcal{U}}$$

- Let $\mathcal{F}$ be the family of all maps $({}^{\kappa}\mathbf{V})^{\mathfrak{M}}$, and given f and g in $\mathcal{F}$, define

$$f \sim_{\mathcal{U}} g \Longleftrightarrow \{m \in M : f(m) = g(m)\} \in \mathcal{U}.$$

- The universe of $\mathfrak{M}^*_{\mathcal{U}}$ consists of the $\sim_{\mathcal{U}}$ equivalence classes $[f]_{\mathcal{U}}$ of members f of $\mathcal{F}$. The membership relation F on $\mathfrak{M}^*_{\mathcal{U}}$ is defined precisely via

$$\langle [f]_{\mathcal{U}}, [g]_{\mathcal{U}} \rangle \in F \Longleftrightarrow \{m \in M : \mathfrak{M}^*_{\mathcal{U}} \models f(m) \in g(m)\} \in \mathcal{U}.$$

Proof of Part (b) of Keisler-Morley's Theorem, Cont'd

- **Theorem** (Łoś-style theorem). *For any first order formula* $\varphi(x_1, \cdots, x_n)$ *and any sequence* $[f_1]_{\mathcal{U}}, \cdots, [f_n]_{\mathcal{U}}$ *the following two conditions are equivalent:*

1. $\mathfrak{M}^*_{\mathcal{U}} \models \varphi([f_1]_{\mathcal{U}}, \cdots, [f_n]_{\mathcal{U}})$;

2. $\{m \in M : \mathfrak{M}^*_{\mathcal{U}} \models \varphi(f_1(m), \cdots, f_n(m))\} \in \mathcal{U}$.

- **Proposition** *There is a nonprincipal ultrafilter* $\mathcal{U}$ *on* $\mathbb{B}$ *such that for* $f \in \mathcal{F}$ *whose range is bounded in* κ*, there is some* $X \in \mathcal{U}$ *such that the restriction of* f *to* X *is constant.*

Proof of Part (b) of Keisler-Morley's Theorem, Cont'd

- Use the Proposition to build an appropriate ultrafilter on $\mathbb{B}$, and form the ultrapower $\mathfrak{M}^*_{\mathcal{U}}$.

- By the Łoś-style theorem, $\mathfrak{M}^*_{\mathcal{U}}$ is an elementary extension of $\mathfrak{M}$, here we are identifying $[c_a]_{\mathcal{U}}$ with the element $a \in M$, where $c_a : \kappa \to \{a\}$.

- The fact that $\mathcal{U}$ is nonprincipal ensures that $\mathfrak{M}^*_{\mathcal{U}}$ is a proper extension of $\mathfrak{M}$ (since the equivalence class $[id]_{\mathcal{U}}$ of the identity function is not equal to any $[c_a]_{\mathcal{U}}$).

- Moreover, the fact that any function in $\mathcal{F}$ with bounded co-domain is constant on a member of $\mathcal{U}$, can be easily seen to imply that $\mathfrak{M}^*_{\mathcal{U}}$ an fixes every element of κ.

L($Q_{\aleph_1}$) via Keisler-Morely (1)

- $\mathcal{L}(Q)$ is the extension of first order logic obtained by adding a new (unary) quantifier Q.

- *Weak models* of $\mathcal{L}(Q)$ are of the form $(\mathfrak{M}, q)$, where $q \subseteq \mathcal{P}(M)$. The Tarski-style definition of satisfaction for weak-models has the new clause:

  $$(\mathfrak{M}, q) \models Qx\varphi(x) \iff \{m \in M : (\mathfrak{M}, q) \models \varphi(m)\} \in q.$$

- A (strong) model of $\mathcal{L}(Q)$ in the κ-interpretation (where κ is an infinite cardinal) is of the form $(\mathfrak{M}, [M]^{\geq\kappa})$, where κ is an infinite cardinal. Here

  $$[M]^{\geq\kappa} := \{X \subseteq M : |X| \geq \kappa\}.$$

- We shall write Q_κ when Q is interpreted in the κ-interpretation. $Val(\mathcal{L}(Q_\kappa))$ is the set of valid sentences of $\mathcal{L}(Q_\kappa)$.

L($Q_{\aleph_1}$) via Keisler-Morely (2)

- **Theorem** [Mostowski 1957].

1. $Val(\mathcal{L}(Q_{\aleph_0}))$ *is not recursively enumerable.*

2. $\mathcal{L}(Q_{\aleph_0})$ *is not countably compact.*

- **Theorem** [Vaught 1964].

1. $\mathcal{L}(Q_{\aleph_0})$ *is countably compact.*

2. $Val(\mathcal{L}(Q_{\aleph_1}))$ *is recursively enumerable.*

L($Q_{\aleph_1}$) via Keisler-Morely (3)

Outline of Proof of countable compactness of $\mathcal{L}(Q_{\aleph_1})$:

Suppose $\Sigma = \{\sigma_n : n \in \omega\}$ is a countable set of $\mathcal{L}(Q)$-sentences such that every finite subset of Σ has a model in $\aleph_1$-interpretation.

Use compactness for first order logic to get hold of a countable non ω-standard model $\mathfrak{M}$ of "enough set theory" such that there is some model $\mathfrak{A}$ in $\mathfrak{M}$ with all $n \in \omega$,

$$\forall n \in \omega \quad \mathfrak{M} \models \text{``}\mathfrak{A} \models \sigma_n\text{''}.$$

Now use the Keisler-Morely theorem to enlarge $\mathfrak{M}$ to a model $\mathfrak{N}$ of set theory such that $(\aleph_1)^{\mathfrak{N}}$is $\aleph_1$-like.

It is now routine to show that $\mathfrak{A}^{\mathfrak{N}}$ is a model of Σ in the $\aleph_1$-interpretation.

A curious Independence Result

- **Theorem** [Cohen 1971]. *There is a model of ZF with an automorphism of order 2*.

- **Remarks:**

  (1) Every standard model of the extensionality axiom is rigid.

  (2) *It is known that if $\mathfrak{M}$ is a model of ZF plus (either AC, or the "the Leibniz-Myscielski axiom"), and j is an automorphism of $\mathfrak{M}$ that fixes all the ordinals of $\mathfrak{M}$, then j is the identity on $\mathfrak{M}$.*

- Consequently, Cohen's theorem yields a new proof of the independence of the axiom of choice from ZF that necessarily uses non-standard models.

A Theorem of Friedman

- **Theorem** [Friedman, 1973]. *Every countable nonstandard model of ZF is isomorphic to a proper rank initial segment of itself.*

Outline of proof for non ω-standard models:

(1) Suppose $\mathfrak{M}$ is a countable non ω-standard model of ZF, and fix a nonstandard integer H in $\mathfrak{M}$.

(2) For each ordinal α of $\mathfrak{M}$, let

$$T_\alpha := (Th(V_\alpha, \in) \cap \{x \in \omega : x < H\})^{\mathfrak{M}}.$$

Note that $\left(T_\alpha \in 2^H\right)^{\mathfrak{M}}$.

Proof of Friedman's Theorem, Cont'd

(3) Invoking the replacement scheme, there is some $K \in \left(2^H\right)^{\mathfrak{M}}$ such that $\mathfrak{M}$ satisfies "$\{\alpha \in \mathbf{Ord} : T_\alpha = K$ is cofinal in the class of ordinals"$\}$.

(4) By the Keisler-Morely theorem, there is an e.e.e. $\mathfrak{N}$ of $\mathfrak{M}$, and by (3), there is some $\beta \in N \backslash M$ such that $T_\beta = k$.

(5) Since $\mathfrak{M}$ is a non ω-standard model of ZF, any structure in $\mathfrak{M}$ is recursively saturated.

(6) [Folklore] Any two recursively saturated countable models of set theory that are (a) elementary equivalent, and (b) have the same "standard system" are isomorphic.

(7) Therefore $(V_\alpha, \in)^{\mathfrak{M}} \cong \left(V_\beta, \in\right)^{\mathfrak{M}}$ for some $\beta \in M$.

(8) The rest is easy!

A weak fragment of set theory

- $EST(\mathcal{L})$ [Elementary Set Theory] is obtained from the usual axiomatization of $ZFC(\mathcal{L})$ by deleting Power Set and Replacement, and adding $\Delta_0(\mathcal{L})$-Separation.

- GW_0 [Global Well-ordering] is the axiom expressing "$\lhd$ well-orders the universe".

- GW is the strengthening of GW_0 obtained by adding the following two axioms to GW_0:

  (a) $\forall x \forall y (x \in y \rightarrow x \lhd y)$;

  (b) $\forall x \exists y \forall z (z \in y \longleftrightarrow z \lhd x)$.

ZFC+'Reflective' Mahlo Cardinals

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  $\{(\kappa$ is n-Mahlo and $V_\kappa \prec_{\Sigma_n} \mathbf{V}) : n \in \omega\}$.

- **Theorem** [E, 2004]. *The following are equivalent for a model $\mathfrak{M}$ of the language $\mathcal{L} = \{\in, \lhd\}$.*

  (a) $\mathfrak{M} = fix(j)$ *for some* $j \in Aut(\mathfrak{M}^*)$, *where* $\mathfrak{M}^* \models EST(\mathcal{L}) + GW$ *and* $\mathfrak{M}^*$ *end extends* $\mathfrak{M}^*$.

  (b) $\mathfrak{M} \models ZFC + \Phi$.

A KEY EQUIVALENCE

- **Theorem.** *If* $(\mathfrak{M}, \mathcal{A}) \models GBC +$ "$\mathbf{Ord}$ *is weakly compact", then* $\mathfrak{M} \models ZFC + \Phi$.

- **Theorem.** *Every countable recursively saturated model of* $ZFC+\Phi$ *can be expanded to a model of* $GBC+$ "$\mathbf{Ord}$ *is weakly compact".*

- **Corollary.** $GBC +$ "$\mathbf{Ord}$ *is weakly compact" is a conservative extension of* $ZFC+\Phi$.

Large Cardinals and Automorphisms

- Suppose M is an $\lhd$-initial segment of $\mathfrak{M}^* :=(M^*, E, <)$. We define:

$$SSy(\mathfrak{M}^*, M) = \{a_E \cap M : a \in M^*\},$$

where $a_E = \{x \in M^* : xEa\}$.

- **Theorem.** *If j is an automorphism of a model $\mathfrak{M}^* = (M^*, E, <)$ of*

$$EST(\{\in, \lhd\}) + GW$$

*whose fixed point set M is a $\lhd$-initial segment of $\mathfrak{M}^*$, and $\mathcal{A} := SSy(\mathfrak{M}^*, M)$, then $(\mathfrak{M}, \mathcal{A}) \models GBC +$ "**Ord** is weakly compact".*