WHAT NFU KNOWS ABOUT CANTORIAN OBJECTS

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BIRTH OF NFU

- Jensen's variant NFU of NF is obtained by modifying the extensionality axiom so as to allow *urelements*.

- $NFU^+ := NFU$ + Infinity + Choice.

- $NFU^- := NFU$ + "V is finite" + Choice.

- **Theorem** (Jensen, 1968) *If (a fragment of) Zermelo set theory is consistent, then so are NFU^+ and NFU^-. Moreover, if ZF has an α-standard model, then so does NFU^+.*

NFU and Orthodox Set Theory (1)

- $EST(\mathcal{L})$ [Elementary Set Theory] is obtained from $ZFC(\mathcal{L})$ by deleting Power Set and Replacement, and adding $\Delta_0(\mathcal{L})$-Separation. More explicitly, it consists of Extensionality, Foundation (every nonempty set has an $\in$-minimal member), Pairs, Union, Infinity, Choice, and $\Delta_0(\mathcal{L})$-Separation.

- GW_0 [Global Well-ordering] is the axiom in the language $\mathcal{L} = \{\in, \lhd\}$, expressing "$\lhd$ well-orders the universe".

- GW is the strengthening of GW obtained by adding the following two axioms to GW_0:

  (a) $\forall x \forall y (x \in y \rightarrow x \lhd y)$;

  (b) $\forall x \exists y \forall z (z \in y \longleftrightarrow z \lhd x)$.

- $ZBQC = EST +$ Power Set.

NFU$^+$ and Orthodox Set Theory (2)

- Jensen's method, as refined by Boffa, shows that one can construct a model of NFU^+ starting from a model $\mathfrak{M}$ of $ZBQC$ that has an automorphism j with $j(\kappa) \geq (2^\kappa)^{\mathfrak{M}}$ for some infinite cardinal κ of $\mathfrak{M}$.

- In the other direction, Hinnion and later, Holmes showed that in NFU^+ one can interpret (1) a 'Zermelian structure' Z that satisfies $ZFC \setminus$ {Power Set}, and (2) a nontrivial endomorphism k of Z onto a proper initial segment of Z.

- The endomorphism k can be used to "unravel" Z to a model $\overline{Z}$ of $ZBQC$ that has a nontrivial automorphism j.

Large Cardinals and NFU (1)

- X is *Cantorian* if there exists a bijection between X and the set of its singletons $UCS(X) := \{\{x\} : x \in X\}$.

- X is *strongly Cantorian* if the graph of the "obvious" bijection $x \longmapsto \{x\}$ between X and $UCS(X)$ forms a set.

- Cantorian elements of models of NFU correspond to fixed points of automorphisms of models of ZF.

Large Cardinals and NFU (2)

- Holmes introduced an extension $NFUM^+$ of NFU^+ which imposes powerful closure conditions on the strongly Cantorian parts of Z, and showed the equiconsistency of $NFUM^+$ and KMC plus "the class of ordinals is a measurable cardinal".

- $NFUM^+$ has two distinguished fragments: $NFUA^+$ and $NFUB^+$.

- $NFUA^{\pm} := NFU^{\pm}$ plus "every Cantorian set is strongly Cantorian".

Large Cardinals and NFU (3)

- Solovay showed:

  (1) $NFUA^+$ is equiconsistent with $ZFC +$ $\{$"there is an n-Mahlo cardinal": $n \in \omega\}$.

  (2) $NFUB^+$ is equiconsistent with KM plus "the class of ordinals is weakly compact".

- **Question:** What does $NFUA^+$ know about the Cantorian part CZ of Z?

- **Answer** (Solovay-Holmes): At least ZFC plus

$\{$"there is an n-Mahlo cardinal": $n \in \omega\}$.

Large Cardinals and Automorphisms (1)

- Let Φ be

$$\{\exists\kappa(\kappa \text{ is } n\text{-Mahlo and } V_\kappa \prec_{\Sigma_n} \mathbf{V}) : n \in \omega\}.$$

- Over ZFC, Φ is stronger than, but equiconsistent with

$$\{\text{“there is an } n\text{-Mahlo cardinal''}: n \in \omega\}.$$

- **Theorem.** *Suppose T is a consistent completion of $ZFC+\Phi$. There is a model $\mathfrak{M}$ of $T+ZF(\lhd)+GW$ such that $\mathfrak{M}$ has a proper e.e.e. $\mathfrak{M}^*$ that possesses an automorphism whose fixed point set is M.*

Large Cardinals and Automorphisms (2)

- **Theorem.** GBC + "**Ord** *is weakly compact*" *is a conservative extension of* $ZFC+\Phi$.

- Suppose M is an $\lhd$-initial segment of $\mathfrak{M}^* := (M^*, E, <)$. We define:

$$SSy(\mathfrak{M}^*, M) = \{a_E \cap M : a \in M^*\},$$

where $a_E = \{x \in M^* : xEa\}$.

- **Theorem.** *If* j *is an automorphism of a model* $\mathfrak{M}^* = (M^*, E, <)$ *of* $EST(\{\in, \lhd\}) + GW$ *whose fixed point set* M *is a* $\lhd$*-initial segment of* $\mathfrak{M}^*$*, and* $\mathcal{A} := SSy(\mathfrak{M}^*, M)$*, then* $(\mathfrak{M}, \mathcal{A}) \models GBC +$ "**Ord** *is weakly compact*".

- **Corollary.** *What* $NFUA^+$ *knows about* CZ *is precisely* $ZFC + \Phi$.

Strong Cuts (1)

- M is a *strong* $\lhd$*-cut* of $\mathfrak{M}^*$, if M is a $\lhd$-cut of $\mathfrak{M}^*$ and for each function $f \in M^*$ whose domain includes M, there is some s in M, such that for all $m \in M$,

$$f(m) \notin M \quad \text{iff} \quad s \lhd f(m).$$

- Suppose $\mathfrak{M}$ is a strong $\lhd$-cut of $\mathfrak{M}^* = (M^*, E, <)$ *of* $EST(\{\in, \lhd\})+GW$ *and* $\mathcal{A} := SSy(\mathfrak{M}^*, M)$.

- *Let* $\mathcal{L}^* = \{\in\} \cup \{S : S \in \mathcal{A}\}$.

- For every $\mathcal{L}^*$-formula $\varphi(\overrightarrow{x}, \overrightarrow{S})$,with free variables $\overrightarrow{x}$ and parameters $\overrightarrow{S}$ from $\mathcal{A}$, there is some $\Delta_0(\mathcal{L})$-formula $\theta_\varphi(\overrightarrow{x}, \overrightarrow{b})$, where $\overrightarrow{b}$ is a sequence of parameters from N, such that for all sequences $\overrightarrow{a}$ of elements of M

$$(\mathfrak{M}, S)_{S \in \mathcal{A}} \models \varphi(\overrightarrow{a}, \overrightarrow{S}) \quad \textit{iff} \quad \mathfrak{M}^* \models \theta_\varphi(\overrightarrow{a}, \overrightarrow{b}).$$

On NFUB$^+$

- $NFUB^+$ is an extension of $NFUA^+$, obtained by adding a scheme that ensures that the intersection of every definable class with CZ is coded by some set.

- Holmes has shown that $NFUB^+$ canonically interprets KMC plus "**Ord** is weakly compact".

- Solovay constructed a model of $NFUB^+$ from a model of KMC plus "**Ord** is weakly compact" in which "$\mathbf{V} = \mathbf{L}$".

- We can show that what $NFUB^+$ knows about its canonical Kelley-Morse model is precisely:

KMC plus "**Ord** is weakly compact" plus
Dependent Choice Scheme

WHAT ABOUT NFU$^-$ AND ITS EXTENSIONS?

- Solovay has shown

(1) ($I\Delta_0$+ *Superexp*) $\vdash$

$$\text{Con}(NFU^-) \iff \text{Con}(I\Delta_0 + Exp).$$

(2) ($I\Delta_0$+ *Exp*) + Con($I\Delta_0 + Exp$) $\nvdash$

$$\text{Con}(NFU^-).$$

On NFUA$^-$

- **Theorem** *The following two conditions are equivalent for any model $\mathfrak{M}$ of the language of arithmetic:*

  (a) $\mathfrak{M}$ *satisfies* PA

  (b) $\mathfrak{M} = fix(j)$ *for some nontrivial automorphism* j *of an end extension* $\mathfrak{N}$ *of* $\mathfrak{M}$ *that satisfies* $I\Delta_0$.

- **Key Lemma.** *If* $\mathfrak{M} \models I\Delta_0$ *and* $j \in Aut(\mathfrak{M})$ *with* $fix(j) \subsetneq_e M$, *then* $fix(j)$ *is a strong cut of* $\mathfrak{M}$.

- **Theorem.** *What* $NFUA^-$ *knows about Cantorian arithmetic is precisely* PA.

NFUA$^-$, ACA$_0$, and VA

- For a cut I of $\mathfrak{M}$, $SSy(\mathfrak{M}, I)$ is the collection of subsets of I of the form $I \cap X$, where X is a coded subset of M.

- (Visser Arithmetic) $VA := I\Delta_0 +$ "j is a nontrivial automorphism whose fixed point set is downward closed".

- **Theorem.** *ACA_0 is faithfully interpretable in VA, and VA is faithfully interpretable in $NFUA^-$.*

ON NFUB$^-$

- Z_2 is second order arithmetic (also known as *analysis*), and DC is the scheme of *Dependent Choice*.

- **Theorem.** *$NFUB^-$ canonically interprets a model of $Z_2 +$ DC. Moreover, every countable model of $Z_2 +$ DC is isomorphic to the canonical model of analysis of some model of $NFUB^-$.*

- **Corollary.** *What $NFUB^-$ knows about analysis is precisely $Z_2 +$ DC.*

AMENABLE AUTOMORPHISMS

- For an initial segment M of $\mathfrak{M}^*$, $SSy(\mathfrak{M}^*, M)$ is the collection of subsets of M of the form $M \cap X$, where X is a subset of M^* that is codes in $\mathfrak{M}^*$.

- $j \in Aut(\mathfrak{M}^*)$ is M-*amenable* if the fixed point set of j is precisely M, and for every formula $\varphi(x, j)$ in the language $\mathcal{L}_A \cup \{j\}$, possibly with suppressed parameters from M^*,

$$\{a \in M : (\mathfrak{M}^*, j) \models \varphi(a, j)\} \in SSy(\mathfrak{M}^*, M).$$

Theorem. *Suppose* $(\mathfrak{M}, \mathcal{A})$ *is a countable model of* $Z_2 + DC$. *There exists an e.e.e.* $\mathfrak{M}^*$ *of* $\mathfrak{M}$ *that has an* M*-amenable automorphism* j *such that* $SSy(\mathfrak{M}^*, M) = \mathcal{A}$.

ON LONGEST INITIAL SEGMENTS OF FIXED POINTS

- For a model $\mathfrak{M}$ of arithmetic and $j \in Aut(\mathfrak{M})$,

$$I_{fix}(j) := \{m \in M : \forall x \le m(j(x) = x)\}.$$

- **Theorem**. *For countable $\mathfrak{M}$, $\mathfrak{M}$ satisfies $I\Delta_0 + B\Sigma_1 + Exp$* iff *$\mathfrak{M} = I_{fix}(j)$ for some nontrivial automorphism j of an end extension $\mathfrak{N}$ of $\mathfrak{M}$ that satisfies $I\Delta_0$.*

- $B\Sigma_1$ is the Σ_1-collection scheme consisting of the universal closure of formulae of the form

$$[\forall x < a \ \exists y \ \varphi(x, y)] \to [\exists z \ \forall x < a \ \exists y < z \ \varphi(x, y)],$$

where φ is a Δ_0-formula.

CONSEQUENCES FOR NFU$^-$

- NFU^- knows that strongly cantorian numbers satisfy $I\Delta_0 + B\Sigma_1 + Exp$.

- But Solovay has shown that NFU^- knows more about strongly cantorian numbers.

- **Question.** What is the precise knowledge of NFU^- about strongly cantorian numbers?

- **Question.** What is the precise knowledge of NFU^+ about the strongly cantorian part of Z?