FROM

FRAGMENTS OF ARITHMETIC

TO

LARGE CARDINALS

VIA

QUINE-JENSEN SET THEORY

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Russell's $\{x : x \notin x\}$(1901)

- Russell's (Ramified)Type Theory *RTT (1908)*
- Ramsey's (Simple) Types Theory *TT (1925)*
- Quine's New Foundations *NF (1937)*
- Wright's NeoFregean Arithmetic *FA (1983)*

- The *language* of NF is $\{=, \in\}$.

- The *logic* of NF is classical first order logic.

- The *axioms* of NF are:

(1) Extensionality:

$$\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y.$$

(2) Stratified Comprehension:

For each stratifiable $\varphi(x)$,

"$\{x : \varphi(x)\}$ exists" , i.e.,

$$\exists z \forall t(t \in z \leftrightarrow \varphi(t)).$$

- φ is *stratified* if there is an integer valued function f whose domain is the set of **all** variables occurring in φ, which satisfies the following two requirements:

(1) $f(v) + 1 = f(w)$, whenever $(v \in w)$ is a subformula of φ;

(2) $f(v) = f(w)$, whenever $(v = w)$ is a subformula of φ.

- The formula $x = x$ is stratifiable, so there is a universal set V in NF.

- NF proves that V is a Boolean algebra.

- Cardinals and ordinals are defined in NF in the spirit of Russell and Whitehead, define $card(X)$ as:

  $\{Y$: there is a bijection from X to $Y\}$.

- $Card := \{\lambda : \exists X(\lambda = card(X)\}$ exists in NF.

- Similarly, the set of all ordinals Ord exists in NF.

- What about Cantor's theorem

$$card(\mathcal{P}(X)) > card(X)$$

applied to V itself?

- In the proof of Cantor's Theorem, given $f : X \to \mathcal{P}(X)$, one needs the set

$$\{x \in X : \neg(x \in f(x))\},$$

whose defining equation is not stratifiable, and therefore unavailable!

- $USC(X) = \{\{x\} : x \in X\}$ exists, and in the NF context, Cantor's theorem is reformulated as:

$$card(\mathcal{P}(X)) > card(USC(X))$$

- For a cardinal λ, let $T(\lambda) := card(USC(X))$, where X is some (any) element of λ, and define (in the metatheory)

$$\begin{aligned} \kappa_0 &:= card(V); \\ \kappa_{n+1} &:= T(\kappa_n). \end{aligned}$$

  NF proves:

$$\kappa_0 > \kappa_1 > \cdots > \kappa_n > \cdots$$

- X is *Cantorian* if $card(X) = card(USC(X))$.

- X is *strongly Cantorian* if $\{\langle x, \{x\}\rangle : x \in X\}$ exists.

- Rosser's *AxCount* (Axiom of Counting):

$\mathbb{N}$ is strongly Cantorian.

- $\mathbb{N} := \{card(X) : fin(X)\}$.

- $fin(X)$ says

"there is no injection from X into a proper subset of X".

- NF proves the equivalence of *AxCount* with "all finite sets are Cantorian".

- **Theorem** (Orey, 1964).

$$NF + \textit{AxCount} \vdash \text{Con}(NF).$$

- **Corollary**. $\text{Con}(NF) \Longrightarrow \text{Con}(NF + \neg \textit{AxCount})$.

- **Theorem** (Hailperin, 1944). *NF is finitely axiomatizable, and $NF = NF_6$.*

- **Theorem** (Grishin, 1969). *$NF = NF_4$, and* Con(NF_3).

- **Theorem** (Boffa, 1977).

$$\text{Con}(NF) \Rightarrow NF \neq NF_3.$$

- **Theorem** (Boffa, 1988; Kaye-Forster 1991). *NF is consistent iff there is a model M of a weak fragment (KF) of Zermelo set theory that possess an automorphism j such that for some $m \in M$, M believes*

$$|j(m)| = |\mathcal{P}(m)|.$$

- **Theorem** (Specker, 1960).

  $$\text{Con}(NF) \Longleftrightarrow \text{Con}(TT + \text{Ambiguity}).$$

- TT is formulated within *multisorted* first order logic with countably many sorts

  $$X_0, X_1, \cdots.$$

- The language of TT is

  $$\{\in_0, \in_1, \cdots\} \cup \{=_0, =_1, \cdots\}.$$

- The atomic formulas are of the form

  $$x^n = y^n, \text{ and } x^n \in_n y^{n+1}.$$

- The axioms of TT consist of:

Extensionality:

$$\Big((\forall z^n((z^n \in_n x^{n+1} \leftrightarrow z^n \in_n y^{n+1})\Big) \rightarrow$$

$$x^{n+1} = y^{n+1}$$

and

Comprehension:

$$\exists z^{n+1}(\forall y^n(y^n \in_n z^{n+1} \leftrightarrow \varphi(x^n)).$$

- The ambiguity scheme consists of sentences of the form $\varphi \longleftrightarrow \varphi^+$, where φ^+ is the result of "bumping all types by 1" in φ.

- Quine-Jensen set theory NFU: relax extensionality to allow urelements.

- MacLane set theory *Mac*: Zermelo set theory with Comprehension restricted to Δ_0-formulas.

- $NFU^+ := NFU$ + infinity + choice.

- $NFU^- := NFU$ + "V is finite" + choice.

- **Theorem** (Jensen, 1968). *Let* $NFU^+ :=$ NFU + *Infinity* + *Choice.*

(1) Con (NFU^+) $\Longleftrightarrow$ Con (*Mac*).

(2) Con (PA) $\Rightarrow$ Con (NFU^-).

(3) *If* ZF *has an* ω*-standard model, then* NFU *has an* ω*-standard model.*

Boffa's simplification of Jensen's proof (1988)

- Arrange a model $\mathfrak{M} := (M, E)$ of Mac, and an automorphism j of M such that

(a) For some infinite $\alpha \in M$, $j(\alpha) < \alpha$, and

(b) V_κ exists in M;

- Define E_{new} on $V_\alpha^{\mathfrak{M}}$ by: $x \ E_{new} \ y$ iff

$x \ E_{new} \ y$ iff $j(x) E y$ and $\mathfrak{M} \models y \in V_{j(\alpha)+1}$.

- **Theorem** (Jensen-Boffa-Hinion). NFU^+ *has a model iff there is a model* $\mathfrak{M}$ *of Mac that has an automorphism* j *such that for some infinite ordinal* α *of* $\mathfrak{M}$,

$$\left(2^{|\alpha|}\right)^{\mathfrak{M}} \leq j(\alpha).$$

Solovay's Work (2002, unpublished)

- $(I\Delta_0 +$ *Superexp*$) \vdash$

$$\text{Con}(NFU^-) \iff \text{Con}(I\Delta_0 + Exp).$$

- $(I\Delta_0 +$ *Exp*$) + \text{Con}(I\Delta_0 + Exp) \nvdash$

$$\text{Con}((NFU^-).$$

Modulo Jensen's work, in order to arrange a model of (NFU + "V is finite") it suffices to build a model M of $(I\Delta_0 + Exp)$ with a nontrivial automorphism j such that for some $m \in M$,

$$2^m \leq j(m).$$

- $NFUA^- := NFU^- +$ "every Cantorian set is strongly Cantorian".

- $VA := I\Delta_0 +$ "j is a nontrivial automorphism whose fixed point set is downward closed".

- **Theorem**. *VA can be (faithfully) interpreted in $NFUA^{-\infty}$.*

- **Theorem.** Con(PA) $\Longleftrightarrow$ Con ($NFUA$).

(1) (Solovay) Con(PA) $\Rightarrow$ Con ($NFUA$).

(2) (E) Con($NFUA$) $\Rightarrow$ Con(PA).

(3) (E) *ACA_0 is (faithfully) interpretable into VA. Therefore (1) cannot be established within $I\Delta_0 + Exp$.*

- **Theorem** (E, 2006). *The following two conditions are equivalent for any model* $\mathfrak{M}$ *of the language of arithmetic:*

(a) $\mathfrak{M}$ *satisfies* PA

(b) $\mathfrak{M} = fix(j)$ *for some nontrivial automorphism* j *of an end extension* $\mathfrak{N}$ *of* $\mathfrak{M}$ *that satisfies* $I\Delta_0$.

- For $j \in Aut(\mathfrak{M})$, $I_{fix}(j) := \{m \in M : \forall x \leq m(j(x) = x)\}$.

- **Theorem** (E, 2006). *The following two conditions are equivalent for a countable model $\mathfrak{M}$ of the language of arithmetic:*

(a) $\mathfrak{M}$ *satisfies* $I\Delta_0 + B\Sigma_1 + Exp$.

(b) $\mathfrak{M} = I_{fix}(j)$ *for some nontrivial automorphism* j *of an end extension* $\mathfrak{N}$ *of* $\mathfrak{M}$ *that satisfies* $I\Delta_0$.

Here $B\Sigma_1$ is the Σ_1-collection scheme consisting of the universal closure of formulae of the form

$$[\forall x < a \ \exists y \ \varphi(x, y)] \rightarrow [\exists z \ \forall x < a \ \exists y < z \ \varphi(x, y)],$$

where φ is a Δ_0-formula.

- $NFUA^{+} := NFU^{+} +$ "every Cantorian set is strongly Cantorian".

- Φ_0 is

  $\{$"there is an n-Mahlo cardinal": $n \in \omega\}$.

- **Theorem** (Solovay 1995, unpublished):

  $$\text{Con}(NFUA^{+}) \iff (ZFC + \Phi_0).$$

- $\Phi := \{$"there is an n-Mahlo cardinal κ such that $V_\kappa \prec_n V$": $n \in \omega\}$.

- Φ_0 is weaker than Φ, but ZF proves $\text{Con}(ZF + \Phi_0) \iff \text{Con}(ZF + \Phi)$.

- **Theorem** (E, 2003).

(1) GBC + "Ord is weakly compact" is (faithfully) interpretable in $NFUA^+$.

(2) The first order part of GBC + "Ord is weakly compact" is precisely Φ.

- $ZF(\mathcal{L})$ is the natural extension of Zermelo-Fraenkel set theory ZF in the language $\mathcal{L} = \{\in, \triangleleft\}$.

- GW is the axiom "$\triangleleft$ is a global well-ordering".

- **Theorem** (E, 2003). *Suppose* T *is a consistent completion of* $ZFC+\Phi$. *There is a model* $\mathfrak{M}$ *of* $T+ZF(\mathcal{L})+GW$ *such that* $\mathfrak{M}$ *has a proper elementary end extension* $\mathfrak{N}$ *that possesses an automorphism* j *whose fixed point set is* M*.*

- **Theorem** (E, 2003). *There is a weak fragment* W *of Zermelo-set theory plus* GW *such that if some model* $\mathfrak{N}=(N,\in^{\mathfrak{N}},\lhd^{\mathfrak{N}})$ *of* W *has an automorphism whose fixed point set* M *forms a proper* $\lhd$*-initial segment of* $\mathfrak{N}$, *then*

$$\mathfrak{M}\models ZF(\mathcal{L})+GW+\Phi,$$

where $\mathfrak{M}$ *is the submodel of* $\mathfrak{N}$ *whose universe is* M.