Set Theory and Indiscernibles

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LEIBNIZ'S PRINCIPLE OF IDENTITY OF INDISCERNIBLES

- The principle of *identity of indiscernibles*, formulated by Leibniz (1686), states that no two distinct substances exactly resemble each other.

- Leibniz's principle can be construed as prescribing a logical relationship between *objects* and *properties*: any two distinct objects must differ in at least one property. This suggests a model theoretic interpretation:

- Fix a model $\mathfrak{M} = (M, \cdots)$ in a language $\mathcal{L}$, let the "objects" refer to the elements of M, and the "properties" refer to properties that are $\mathcal{L}$-expressible in $\mathfrak{M}$ via first order formulas with one free variable.

LEIBNIZIAN MODELS

- Let us call a model $\mathfrak{M}$ to be *Leibnizian* iff $\mathfrak{M}$ contains no pair of distinct elements a and b, such that for every first order formula $\varphi(x)$ of $\mathcal{L}$ with precisely one free variable x,

$$\mathfrak{M} \models \varphi(a) \leftrightarrow \varphi(b).$$

- Any pointwise definable model is Leibnizian, e.g., $(\omega, <)$, $(V_\omega, \in)$, and $(L(\omega_1^{CK}), \in)$.

- Any model $\mathfrak{M} = (M, \cdots)$ in a language $\mathcal{L}$ such that $|M| > 2^{|\mathcal{L}| \cdot \aleph_0}$ is *not* Leibnizian.

- Every Leibnizian model is rigid, but *not* vice versa: $(\omega_1, <)$ is rigid but not Leibnizian.

LEIBNIZIAN MODELS, CONT'D

- The field $\mathbb{R}$ of real numbers, and the ring of integers $\mathbb{Z}$ are both Leibnizian, but the field $\mathbb{C}$ of complex numbers is *not.*

- Every Archimedean ordered field is Leibnizian.

- Moreover, Tarski's elimination of quantifiers theorem for real closed fields implies that the *Leibnizian real closed fields are precisely the Archimedean real closed fields*.

- Non-Archimedean Leibnizian ordered fields exist in every infinite cardinality $\leq 2^{\aleph_0}$.

THE LEIBNIZ-MYCIELSKI AXIOM (LM)

- Leibniz's principle cannot be expressed in first order logic, even for countable structures. This is an immediate corollary of Ehrenfeucht-Mostowski's theorem on indiscernibles.

- However, Mycielski (1995) has introduced the following first order axiom (LM) in the language of set theory $\{\in\}$which captures the spirit of Leibniz's principle for models of set theory:

  $$\forall x \forall y\ [x \neq y \to \exists \alpha > \max\{\rho(x), \rho(y)\}$$

  $$Th(V_\alpha, \in, x) \neq Th(V_\alpha, \in, y)].$$

- **Theorem** (Mycielski). *A complete extension T of ZF proves LM iff T has a Leibnizian model.*

LM AS A CHOICE PRINCIPLE

- *Kinna-Wagner Selection Principles* (1955)

  KW_1: For every family $\mathcal{A}$ of sets there is a function f such that

  $\forall x \in \mathcal{A}\ (|x| \geq 2 \rightarrow \emptyset \neq f(x) \subsetneq x)$.

  KW_2: Every set can be injected into the power set of some ordinal.

$$ZF \vdash KW_1 \longleftrightarrow KW_2.$$

- GKW_1: There is a definable (without parameters) map $\mathbf{F}$ such that $\mathbf{F}(x) \subsetneq x)$ for every x with two or more elements.

- GKW_2: There is a definable (without parameters) map $\mathbf{G}$ such that $\mathbf{G}$ injects $\mathbf{V}$ into the class of subsets of $\mathbf{Ord}$".

THE EQUIVALENCE OF LM WITH GLOBAL KM

Theorem. *Suppose M is a model of ZF. The following are equivalent:*

(i) *M satisfies GKW_1.*

(ii) *M satisfies GKW_2.*

(iii) *M satisfies LM.*

COROLLARIES

Corollary. $ZF + LM \vdash KW$.

Corollary. $ZF + \mathbf{V} = \mathbf{OD} \vdash LM$.

Corollary *In the presence of* $ZF + LM$ *there is a parameter free definable global linear ordering of the universe.*

Corollary. $ZF+LM$ proves $GC_{<\omega}$(*global choice for collections of finite sets*).

Corollary. $ZF + LM$ *proves the existence of a definable set of real numbers that is not Lebesgue measurable.*

OPEN QUESTIONS

- **Question 1.** (Abramson and Harrington, 1977). Does every completion T of ZF have an uncountable model without a pair of indiscernible ordinals?

- **Question 2** (Schmerl). Is there a model of set theory with a pair of indiscernibles, but not with a *triple* of indiscernibles?

ANTI-LEIBNIZIAN SYSTEMS

$ZFC(I)$ is a theory in the language $\{\in, \mathbf{I}(x)\}$, where $\mathbf{I}(x)$ is a unary predicate [but we shall write $x \in \mathbf{I}$ instead of $\mathbf{I}(x)$], whose axioms are:

- ZFC + All instances of replacement in the language $\{\in, \mathbf{I}(x)\}$;

- $\mathbf{I}$ is a cofinal subclass of ordinals:

  $(\mathbf{I} \subseteq \mathbf{Ord}) \wedge$

  $\forall x \in \mathbf{Ord}\ \exists y \in \mathbf{Ord}\ (x \in y \in \mathbf{I})$;

- For each n-ary formula $\varphi(v_1, \cdots, v_n)$ in the language $\{\in\}$,

  $\forall x_1 < \cdots < x_n$, $\forall y_1 < \cdots < y_n$ from $\mathbf{I}$

$$\varphi(x_1, \cdots, x_n) \leftrightarrow \varphi(y_1, \cdots, y_n).$$

$ZFC(I)$ AND LARGE CARDINALS

- If κ is a Ramsey cardinal, then $(V_\kappa, \in)$ expands to a model of $ZFC(I)$.

- If $(L_\kappa, \in)$ expands to a model of $ZFC(I)$, and cf$(\kappa) > \omega$, then $0^{\#}$ exists.

- If $0^{\#}$ exists, then $\mathbf{L}$ cannot be expanded to $ZFC(I)$.

- Every well-founded model of $ZFC(I)$ satisfies $0^{\#}$ exists.

THE SYSTEM $ZFC(I^{<\omega})$

- $ZFC(I^{<\omega})$ is a theory in the language
$$\{\in\} \cup \{\mathbf{I}_n(x) : n \in \omega\},$$
where each $\mathbf{I}_n$ is a unary predicate, whose axioms are:

- ZFC + All instances of replacement in the language $\{\in\} \cup \{\mathbf{I}_n(x) : n \in \omega\}$;

- $\mathbf{I}_n$ is a cofinal subclass of ordinals;

- $\mathbf{I}_0$ is a class of indiscernibles for $(\mathbf{V}, \in)$, and for $n \geq 0$, $\mathbf{I}_{n+1}$ is a class of indiscernibles for the structure $(\mathbf{V}, \in, \mathbf{I}_0, \cdots, \mathbf{I}_n)$.

- **Question.** *What are the consequences of $ZFC(I)$ and $ZFC(I^{<\omega})$ in the $\in$-language of set theory?*

THE ANSWER

- **Theorem.** *The following are equivalent for a completion T of ZFC:*

1. *Some model of T expands to a model of $ZFC(I)$.*

2. *Some model of T expands to a model of $ZFC(I^{<\omega})$.*

3. *Some model of T expands to a model of $GBC +$ "$\mathbf{Ord}$ is weakly compact".*

4. *T is a completion of $ZFC + \Phi$.*

- $GBC =$ Gödel-Bernays class theory.

- "$\mathbf{Ord}$ is weakly compact" is the statement "every **Ord**-tree has a branch".

THE CANONICAL SET THEORY

$ZFC + \Phi$

- $\Phi := \{\exists\theta\ (\theta$ is n-Mahlo and $V_\theta \prec_n \mathbf{V}) : n \in \omega\}$

- $\Phi_0 := \{\ \exists\theta\ (\theta$ is n-Mahlo$) : n \in \omega\}$.

- *Over ZF, Φ and Φ_0 are equivalent.*

- **Motto:** Φ *allows infinite set theory to catch up with finite set theory, vis-à-vis Model Theory.*

A KEY EQUIVALENCE

- **Theorem.**

1. *If* $(\mathfrak{M}, \mathcal{A}) \models GBC +$ "**Ord** *is weakly compact*", *then* $\mathfrak{M} \models ZFC + \Phi$.

2. *Every consistent completion of* $ZFC + \Phi$ *has a countable model which has an expansion to a model of* $GBC +$ "**Ord** *is weakly compact*".

[Schmerl-Shelah (1972) $\rightsquigarrow$Kaufmann (1983) $\rightsquigarrow$ E(1987, 2004)]

CONCLUDING CONSIDERATIONS

- If Replacement(I) is weakened to Separation(I) in $ZFC(I)$, while retaining "I is cofinal", then the resulting theory is conservative over ZFC.

- We can strengthen $ZFC(I)$ to $ZFC(I^+)$ with "C is a cub" to ensure that when the indiscernibles are stretched, a model with a *least new ordinal* is obtained.

- $ZFC(I^+)$ turns out to be a conservative extension of a $ZFC + \Psi$, where the scheme Ψ is obtained from Φ by replacing "n-Mahlo" by "n-subtle", i.e., the axioms of Ψ are of the form

$$\text{“}\exists\theta(\theta \text{ is } n\text{-subtle and } V_\theta \prec_n \mathbf{V}\text{”}.$$