Automorphisms, Mahlo Cardinals, and NFU

Ali Enayat

ABSTRACT. In what follows, $ZF(\mathcal{L})$ is the natural extension of Zermelo-Fraenkel set theory ZF in the language $\mathcal{L} = \{\in, \lhd\}$, GW is the axiom "$\lhd$ is a global well-ordering", and $\Phi := \{\varphi_n : n \in \omega\}$, where φ_n asserts the existence of a Σ_n-reflecting n-Mahlo cardinal. Our principal results are Theorem A, and its "reversal" Theorem B below.

Theorem A. *Suppose T is a consistent completion of $ZFC+ \Phi$. There is a model $\mathfrak{M}$ of $T + ZF(\mathcal{L}) + GW$ such that $\mathfrak{M}$ has a proper elementary end extension $\mathfrak{N}$ that possesses an automorphism j whose fixed point set is M.*

Theorem B. *There is a weak fragment W of Zermelo-set theory plus GW such that if some model $\mathfrak{N} = (N, \in^{\mathfrak{N}}, \lhd^{\mathfrak{N}})$ of W has an automorphism whose fixed point set M forms a proper $\lhd$-initial segment of $\mathfrak{N}$, then*

$$\mathfrak{M} \models ZF(\mathcal{L}) + GW + \Phi,$$

where $\mathfrak{M}$ is the submodel of $\mathfrak{N}$ whose universe is M.

We also explain how Theorems A and B can be used to fine tune the work of Solovay and Holmes concerning the relationship between Mahlo cardinals and the extension $NFUA$ of the Quine-Jensen system NFU.

1. INTRODUCTION AND PRELIMINARIES

In this paper we establish an intimate relationship between Mahlo cardinals and automorphisms of models of set theory. The models of set theory involved here are not well-founded since a routine argument by transfinite induction reveals that a well-founded model satisfying the extensionality axiom only admits the trivial automorphism. The source of our inspiration was Solovay's calibration of the consistency strength of the unorthodox system of set theory $NFUA$, but our results can also be motivated in an orthodox fashion by comparing the behavior of automorphisms of models of ZFC (Zermelo-Fraenkel set theory with the axiom of choice), with those of ZFC's sister theory PA (Peano Arithmetic). Our point of departure in this comparative context is Gaifman's seminal work [G] on the model theory of PA. Gaifman refined the MacDowell-Specker method [MS] of building elementary end extensions by introducing the powerful machinery of *minimal end extension types*, which can be used to produce a variety of models of PA with special properties. In particular, one can use minimal end extension types to establish that *every model $\mathfrak{M} = (M, +, \cdot)$ of PA has a proper elementary end extension $\mathfrak{N}$*

2000 *Mathematics Subject Classification.* Primary 03C62, 03E55; Secondary 03C50, 03H99.
Key words and phrases. Automorphisms, models of set theory, Mahlo cardinals, NFU.

that possesses an automorphism j whose fixed point set is precisely M (as observed by Kossak, this is a corollary of [G, Theorems 4.9, 4.10 and 4.11]). Therefore,

Theorem 1.1. *Every consistent completion of PA has a model $\mathfrak{N}$ that possesses a nontrivial automorphism whose fixed point set is a proper initial segment of $\mathfrak{N}$.*

A natural question is whether there is an analogue of Theorem 1.1 for ZFC. We answer this question by showing, in Theorem A (Section 3) and Theorem B (Section 4), that such an analogue exists *precisely* for consistent completions of the theory $ZFC + \Phi$, where

$$\Phi := \{\exists\kappa(\kappa \text{ is } n\text{-Mahlo and } V_\kappa \text{ is a } \Sigma_n\text{-elementary submodel of } \mathbf{V}) : n \in \omega\}.$$

(intuitively, Φ says that the class of ordinals behaves like an ω-Mahlo cardinal). The prehistory of the scheme Φ goes back to the groundbreaking work of Schmerl and Shelah [SS] on the Hanf-number computation of power-like models in terms of Mahlo cardinals, but Φ itself was first explicitly introduced and studied by Kaufmann [Ka] and the author [E-1, 2] in the context of identifying the right "completion" of ZFC which resembles PA from the point of view of model theory. For example, every consistent completion of $ZFC + \Phi$ has a κ-like model for any uncountable cardinal κ ([Kau, Theorem 3.10], [E-1, Theorem 3.8]). Moreover, if $\aleph_2$ has the tree property, then every $\aleph_2$-like model of ZFC already satisfies Φ [E-3, Theorem 4.8]. By [E-1, Remark 1.6] there is no analogue of minimal end extension types for models of ZFC in any *finite* alphabet extending $\{\in\}$, but our work here shows that, in contrast, there is such an analogue for the enriched theory $ZFC(\mathcal{L}) + \Phi(\mathcal{L})$, where $\mathcal{L}$ is an expansion of $\{\in\}$ by *infinitely* many predicates. This provides further a posteriori evidence for the maxim:

$$ZFC + \Phi \text{ imitates } PA\text{, vis-à-vis model theory.}$$

In Section 5 we use the results obtained about automorphisms of models of ZFC to gain insight into models of the set theoretical system $NFUA$ (an extension of the Quine-Jensen system NFU of set theory). In particular, Theorems A and B allow us to pinpoint the first order theories of the so-called Cantorian initial segments $CZ^{\mathfrak{A}}$ of Zermelian structures $Z^{\mathfrak{A}}$ arising from models $\mathfrak{A}$ of $NFUA$, which in turn lead to a new proof of the following result:

Theorem 1.2.[1] (Solovay, 1995) *The following theories are equiconsistent*:

$T_1 := ZFC+$ { *"there is an n-Mahlo cardinal"*: $n \in \omega$},
$T_2 := NFUA$.

There is also an arithmetical counterpart to our set theoretical story. Recent work of Solovay and the author has revealed a close relationship between classical arithmetical theories (ranging from a fragment of PA to second order arithmetic) and natural extensions of the theory obtained by adding the *negation* of the axiom of infinity to NFU. The picture that emerges from this work shows that NFU provides a robust framework for the formulation of elegant theories whose consistency strength varies from the tangible territory of the fragment $I\text{-}\Delta_0 + Exp$ of PA, all the way up to the heights of systems of set theory with large cardinals. In light of these developments, the Quine-Jensen paradigm emerges as a fascinating

[1]This result is announced (without proof) in Solovay's online manuscript [So].

foundational alternative that not only offers a coherent rival account of set theory[2], but also unifies seemingly unrelated systems of classical arithmetic and set theory.

I am grateful to Robert Solovay for pointing the way in [So], enduring my rough drafts, and providing precious advice. I am also indebted to Randall Holmes for helpful discussions about NFU, and to the anonymous referee for suggested improvements.

Preliminaries: Theories and Models

A. Theories

- For a *relational language* $\mathcal{L}$, $\Sigma_n(\mathcal{L})$, $\Pi_n(\mathcal{L})$, and $\Delta_n(\mathcal{L})$ are sets of first order formulas built up from $\mathcal{L}$ that (respectively) belong to the Σ_n-level, Π_n-level, and Δ_n-level of the well-known Lévy hierarchy. We use $\Sigma_\infty(\mathcal{L})$ for $\bigcup\limits_{n\in\omega} \Sigma_n(\mathcal{L})$.
- For a class Γ of formulas, Γ-*Separation* consists of formulas of the form
$$\forall a\exists b\forall x(x\in b \longleftrightarrow x\in a\wedge\varphi(x)),$$
for all formulas $\varphi\in\Gamma$ in which b does not occur free.
- For a class Γ of formulas, Γ-*Replacement* consists of formulas of the form
$$[\forall x\in a\exists! y\varphi(x,y)]\rightarrow[\exists b\forall y(y\in b\leftrightarrow\exists x\in a\wedge\varphi(x,y))],$$
for all formulas $\varphi\in\Gamma$ in which b does not occur free.
- $ZF(\mathcal{L})$ is ZF augmented by $\Sigma_\infty(\mathcal{L})$-Replacement, where $\mathcal{L}\supseteq\{\in\}$.
- $EST(\mathcal{L})$ [Elementary Set Theory] is obtained from the usual axiomatization of $ZFC(\mathcal{L})$ by deleting Power Set and $\Sigma_\infty(\mathcal{L})$-Replacement, and adding $\Delta_0(\mathcal{L})$-Separation. More explicitly, it consists of Extensionality, Foundation (every nonempty set has an $\in$-minimal member), Pairs, Union, Infinity, Choice, and $\Delta_0(\mathcal{L})$-Separation. The theory KP + Infinity (where KP is Kripke-Platek set theory as in [Bar]) is a significantly stronger theory than $EST\{\in\}$ since KP includes Σ_∞-Foundation, and Δ_0-Replacement.
- GW [Global Well-ordering] is the axiom in the language $\mathcal{L}=\{\in,\lhd\}$, expressing "$\lhd$ well-orders the universe". More explicitly, GW is the conjunction of the axioms "$\lhd$ is a total order" and "every nonempty set has a $\lhd$-least element". A well-known forcing argument shows that every countable model of ZFC expands to a model of $ZF\{\in,\lhd\}+GW$ (see, e.g. [Fe]).
- GW^* is the strengthening of GW obtained by adding the following two axioms to GW:

  (a) $\forall x\forall y(x\in y\rightarrow x\lhd y)$;
  (b) $\forall x\exists y\forall z(z\in y\longleftrightarrow z\lhd x)$.

  It is easy to see that every model of $ZF(\{\in,\lhd\})+GW(\lhd)$ can be expanded to a model of $ZF(\{\in,\blacktriangleleft\})+GW^*(\blacktriangleleft)$ since the desired ordering $\blacktriangleleft$ satisfying GW^* is defined by:
$$x\blacktriangleleft y\Leftrightarrow[(x\lhd y \text{ and } \rho(x)=\rho(y)) \text{ or } \rho(x)\in\rho(y)],$$

[2] Rosser's classic text [Ro] offers a development of mathematics within NF. See also Forster's [Fo] and Holmes's [Ho-1] for samples of recent work in this area.

where ρ is the usual ordinal-valued rank function. However, over weaker set theories GW^* might be stronger than GW. Also, note that if α is an admissible ordinal, or a limit of admissible ordinals, and L_α is the α-th level of the constructible hierarchy, then by Σ_1-reflection [Bar, Theorem 4.3],

$$(L_\alpha, \in, <_{L_\alpha}) \models EST(\{\in, \lhd\}) + GW^*,$$

where $<_{L_\alpha}$ is Gödel's canonical well-ordering of L_α.

- For a finite relational language $\mathcal{L}$, $\Phi(\mathcal{L})$ is the scheme $\{\varphi_n(\mathcal{L}) : n \in \omega\}$, where $\varphi_n(\mathcal{L})$ is the formula expressing:

$$\exists\kappa(\kappa \text{ } is \text{ } n\text{-Mahlo and } (V_\kappa, \in, \cdots) \prec_{\Sigma_n(\mathcal{L})} (\mathbf{V}, \in, \cdots).$$

- GBC is the Gödel-Bernays theory[3] of classes GB with the *class form* of the axiom of choice.

B. Models

- Models of set theory are of the form $\mathfrak{M} = (M, E, \cdots)$ and $\mathfrak{N} = (N, F, \cdots)$, where $E = \in^{\mathfrak{M}}$, and $F = \in^{\mathfrak{N}}$.
- $\mathfrak{M}$ is a *submodel* of $\mathfrak{N}$, written $\mathfrak{M} \subseteq \mathfrak{N}$, if $M \subseteq N$ and $E = F \cap M^2$.
- For $\mathfrak{M} = (M, E, \cdots)$, and $a \in M$, $a_E = \{b \in M : bEa\}$.
- $\mathfrak{N}$ *end extends* $\mathfrak{M}$ (equivalently: $\mathfrak{M}$ is an *initial* submodel of $\mathfrak{N}$), written $\mathfrak{M} \subseteq_e \mathfrak{N}$, if $\mathfrak{M}$ is a submodel of $\mathfrak{N}$ and for every $a \in M, a_E = a_F$.
- We abbreviate "elementary end extension" by "e.e.e.". It is well-known that if $\mathfrak{N}$ is an e.e.e. of a model $\mathfrak{M}$ of ZF, then $\mathfrak{N}$ is a *rank extension* of $\mathfrak{M}$, i.e., whenever $a \in M$ and $b \in N \backslash M$, then $\mathfrak{N} \models \rho(a) \in \rho(b)$.
- $\mathfrak{N}$ is said to be a *conservative extension* of $\mathfrak{M}$, if $\mathfrak{M} \subseteq \mathfrak{N}$ and for all parametrically definable subsets X of $\mathfrak{N}$, $X \cap M$ is definable in $\mathfrak{M}$. For $\mathfrak{M} \prec_e \mathfrak{N} \models ZF$, this is equivalent to saying that for all $c \in N \backslash M, c_F \cap M$ is parametrically definable in $\mathfrak{M}$.
- Suppose $\mathcal{L} \supseteq \{\in\}$ is a relational language. Given structures $\mathfrak{M}$ and $\mathfrak{N}$ of the same language $\mathcal{L}$, $\mathfrak{M} \prec_{\Sigma_n(\mathcal{L})} \mathfrak{N}$ means that $\mathfrak{M}$ is a $\Sigma_n(\mathcal{L})$-elementary submodel of $\mathfrak{N}$, i.e., every formula of $\Sigma_n(\mathcal{L})$ is absolute in the passage between $\mathfrak{M}$ and $\mathfrak{N}$. For finite $\mathcal{L}$, $\Sigma_n(\mathcal{L})$-truth is $\Sigma_n(\mathcal{L})$-definable within $ZF(\mathcal{L})$ for $n \geq 1$ [J, Section 14]. Therefore, for every finite language $\mathcal{L}$, and every natural number n there is a single $\mathcal{L}$-formula $\varphi_n(x)$ such that for all models $\mathfrak{M}$ of $ZF(\mathcal{L})$, and all $\alpha \in \mathbf{Ord}^{\mathfrak{M}}$, $\mathfrak{M} \models \varphi(\alpha)$ iff $(V_\alpha, \in, \cdots)^{\mathfrak{M}} \prec_{\Sigma_n(\mathcal{L})} \mathfrak{M}$.
- Models of GB can be written in the two-sorted form $(\mathfrak{M}, \mathcal{A})$, where $\mathfrak{M}$ is a model of ZF, and $\mathcal{A}$ is a family of subsets of M. Since coding of sequences is available in GB, we shall use expressions such as "$f \in \mathcal{A}$", where f is a function, as a substitute for the precise but lengthier expression "the canonical code of f is in $\mathcal{A}$". $S \in \mathcal{A}$ is said to form a *proper class* if there is no $c \in M$ such that $c_E = S$, else S is said to form a *set*. It is well-known that for $\mathcal{A} \subseteq \mathcal{P}(M)$, and $\mathfrak{M} \models ZF$, $(\mathfrak{M}, \mathcal{A}) \models GB$ iff $(\mathfrak{M}, S)_{S \in \mathcal{A}} \models ZF(\mathcal{L})$, where $\mathcal{L} = \{\in\} \cup \{S : S \in \mathcal{A}\}$.

---

[3]We have followed Mostowski's lead in our adoption of the appellation GB, but Jech's text [Jec] dubs this theory BG. To make matters more confusing, the same theory is also known in the literature as VNB (*von Neumann-Bernays)* and NBG (*von Neumann-Bernays-Gödel)*.

2. THE STRENGTH OF "ORD IS WEAKLY COMPACT"

In this section we probe the strength of the assertion "**Ord** is weakly compact" within GBC. Here "**Ord** is weakly compact" is the statement in class theory asserting that every **Ord**-tree has a branch[4]. **Ord**-trees are defined in analogy with the familiar notion of κ-trees in infinite combinatorics: $(\tau, <_\tau)$ is an **Ord**-tree, if $(\tau, <_\tau)$ is a well-founded tree of height **Ord** such that the collection of nodes of any prescribed ordinal rank form a set. The principal results of this section are Theorems 2.1 and 2.2. Theorem 2.1 reveals an unexpected equivalence between the class theory GBC+ "**Ord** is weakly compact" and the set theory $ZFC + \Phi$, which can be paraphrased as:

$$\Phi/ZFC \sim \text{“}\mathbf{Ord}\text{ is weakly compact”}/GBC.$$

Theorem 2.2, on the other hand, establishes a combinatorial theorem that plays a vital role in the proof of Theorem A.

It is a classical theorem of ZFC set theory that if κ is a weakly compact cardinal, then κ is κ-Mahlo. In contrast, Corollary 2.1.1 of Theorem 2.1 implies that the set theoretical consequences of GBC + "**Ord** is weakly compact" are precisely the same as those of $ZFC + \Phi$. Consequently, even though the theory GBC + "**Ord** is weakly compact" can prove the family of statements of the form "**Ord** is n-Mahlo" for standard natural numbers n, it is unable to prove "**Ord** is ω-Mahlo". To see this, note that

$$GBC + \text{“}\mathbf{Ord}\text{ is }\omega\text{-}Mahlo\text{”} \vdash \text{“}ZFC + \Phi\text{ is finitely satisfiable”}.$$

Therefore, since Corollary 2.1.1 is a theorem of PA,

$$GBC + \text{“}\mathbf{Ord}\text{ is }\omega\text{-}Mahlo\text{”} \vdash \mathrm{Con}(GBC + \text{“}\mathbf{Ord}\text{ is weakly compact”}).$$

So, by Gödel's second incompleteness theorem,

$$GBC + \text{“}\mathbf{Ord}\text{ is weakly compact”} \nvdash \text{“}\mathbf{Ord}\text{ is }\omega\text{-Mahlo”}.$$

Theorem 2.1.
1. *If* $(\mathfrak{M}, \mathcal{A}) \models GBC +$ "**Ord** *is weakly compact", then* $\mathfrak{M} \models ZFC + \Phi$.
2. *Every consistent completion of* $ZFC + \Phi$ *has a countable model that has an expansion to a model of* $GBC +$ "**Ord** *is weakly compact".*

Before proceeding with the proof of Theorem 2.1, let us derive two corollaries.

Corollary 2.1.1. *The following are equivalent for a sentence* ψ *in the language of set theory:*

1. $ZFC + \Phi \vdash \varphi$.
2. $GBC +$ "**Ord** *is weakly compact*" $\vdash \psi^{\mathbf{V}}$.

Proof. This is an immediate consequence of coupling Theorem 2.1 with the completeness theorem for first order logic. □

[4]The discriminating reader might prefer "**Ord** has the tree property" to express this property. But all is well, since one can use the methods of this section to show that within GBC the statement "Every **Ord**-tree has a branch" is equivalent to the statement "If T is a proper class of sentences in the infinitary logic $L_{\infty,\infty}$ (allowing less than **Ord**-many conjunctions and strings of quantifiers) every subset of which has a model, then T has a model".

Corollary 2.1.2. *Suppose $GBC+$ "$\mathbf{Ord}$ is weakly compact" is consistent. If ψ is a sentence in the language of set theory satisfying either*
1. $ZF \vdash$ "*ψ holds in $\mathbf{L}$*", *or*
2. $ZFC \vdash$ "*for some poset $\mathbb{P}$, $1_{\mathbb{P}} \Vdash \psi$*",
then there is a model of $GBC+$ "$\mathbf{Ord}$ is weakly compact" $+\ \psi^{\mathbf{V}}$.

Proof. This follows from Corollary 2.1.1 and the observation that if $\mathfrak{M}$ satisfies $ZFC+\Phi$, then (1) Φ holds in the constructible universe $\mathbf{L}^{\mathfrak{M}}$ of $\mathfrak{M}$, and (2) Φ holds in every $\mathbb{P}$-generic extension of $\mathfrak{M}$, where $\mathbb{P} \in M$. (1) is easy to verify. (2) is a consequence of the preservation of both (a) the n-Mahlo property of a cardinal κ, and (b) the property "$V_\kappa \prec_{\Sigma_n} \mathbf{V}$", in $\mathbb{P}$-generic extensions satisfying $\mathbb{P} \in V_\kappa$. (a) is established along the lines of the proof of [Kan, Proposition 10.13]; (b) follows from a standard forcing argument. □

The proof of part (1) of Theorem 2.1 is based on the methods introduced in [E-2, E-3]. Notice that the proof would have been routine if GBC were to be replaced by the stronger theory KMC (Kelley-Morse theory of classes with global choice), since the usual ZFC-proof establishing the κ-Mahlo property of a weakly compact cardinal κ can be conveniently implemented within KMC. We need the following refinement of the KMC-proof, tailor made for GBC.

Lemma 2.1.2. *Suppose $(\mathfrak{M}, \mathcal{A})$ is a model of $GBC+$ "$\mathbf{Ord}$ is weakly compact" and $S \in \mathcal{A}$. For every natural number n, there is a Σ_n-e.e.e. $(\mathfrak{N}, S^*)$ of $(\mathfrak{M}, S)$ such that $\mathbf{Ord}^{\mathfrak{N}} \backslash M$ has a minimum element, and $\{c_F \cap M : c \in N\} \subseteq \mathcal{A}$, where $F = \in^{\mathfrak{N}}$.*

Proof. By global choice, there is some $\lhd \in \mathcal{A}$ such that $(M, \lhd) \models GW$. Given any $S \in \mathcal{A}$, $(\mathfrak{M}, f, S) \models ZF(\{\in, \lhd, S\})$, and therefore, as shown in [E-3, Section 3], for every natural number n there is a tree τ_n such that (i) through (iii) below hold:
(i) τ_n is definable in $(\mathfrak{M}, \lhd, S)$;
(ii) $(\mathfrak{M}, \lhd, S) \models$ "τ_n is an $\mathbf{Ord}$-tree";
(iii) If τ_n has a branch B, then there is a model $(\mathfrak{N}, S^*)$, and an embedding

$$j : (\mathfrak{M}, S) \rightarrow (\mathfrak{N}, S^*),$$

where both $(\mathfrak{N}, S^*)$ and j are definable within $(\mathfrak{M}, \lhd, S, B)$, such that

$$(\mathfrak{M}, \lhd, S, B) \models \text{"}j(\mathfrak{M}, S) \text{ is a } \Sigma_n\text{-elementary initial segment of } (\mathfrak{N}, S^*)\text{"}.$$

(i) through (iii) together imply that for every natural number n,
(iv) $(\mathfrak{M}, \mathcal{A}) \models$ "$(\mathfrak{M}, S)$ has a Σ_n-e.e.e. $(\mathfrak{N}, S^*)$".

It remains to show that $(\mathfrak{N}, S^*)$ can be arranged to have a least new ordinal. This is a consequence of the following result, which can be established by adapting the proof of [Kau-2, Lemma 3.2] or [E-2, Theorem 3.3] to the present setting.

Proposition 2.1.3. *Suppose $\mathfrak{M}$ is a model of $ZFC(\mathcal{L})$ for a finite relational language $\mathcal{L}$ such that for some $n \in \omega$, there is a model $\mathfrak{N}$ such that*
(1) $\mathfrak{M} \prec_{e,\Sigma_{n+2}(\mathcal{L})} \mathfrak{N}$, *and*
(2) *for every parametrically $\mathfrak{N}$-definable subset X of N,*

$$(\mathfrak{M}, X \cap M) \models ZF(\mathcal{L} \cup \{S\}),$$

where S is a unary predicate interpreted by $X \cap M$. There is a submodel $\mathfrak{N}_0$ of $\mathfrak{N}$ such that $\mathfrak{M} \prec_{e,\Sigma_n(\mathcal{L})} \mathfrak{N}_0$ and $\mathbf{Ord}^{\mathfrak{N}_0} \backslash M$ has a least element. □

Proof of part 1 of Theorem 2.1. We are now in a position to verify that if $(\mathfrak{M}, \mathcal{A})$ is a model of GBC + "**Ord** is weakly compact", then $\mathfrak{M}$ satisfies Φ. *We shall use an external induction to show that for each standard natural number n, if $\mathfrak{N}$ is a Σ_1-e.e.e. of $\mathfrak{M}$ satisfying*

(✠) $\{c_F \cap M : c \in N\} \subseteq \mathcal{A}$, *where* $F = \in^{\mathfrak{N}}$,

and $\gamma = \min(\mathbf{Ord}^{\mathfrak{N}} \backslash M)$, *then γ is n-Mahlo in $\mathfrak{N}$.* Coupled with Lemma 2.1.2, this shows that Φ holds in $\mathfrak{M}$. Suppose $\mathfrak{N}$ is a Σ_1-e.e.e. of $\mathfrak{M}$ satisfying (✠) with a least new ordinal γ. The case $n = 0$ follows from the simple observation that γ cannot be singular in $\mathfrak{N}$ because any potential singularizing map in $\mathfrak{N}$ is already in $\mathcal{A}$ by (✠), and GB rules out such an entity. Moreover, γ must be a strong limit cardinal because if $\lambda < \gamma$ then $(2^\lambda)^{\mathfrak{M}} = (2^\lambda)^{\mathfrak{N}}$ since "x is the power set of y" is a Π_1-predicate and, of course, $(2^\lambda)^{\mathfrak{M}}$ is a member of $\mathfrak{M}$ and therefore is less than γ. Now assume the inductive hypothesis for some $n \in \omega$, and assume on the contrary that γ is not $(n+1)$-Mahlo in $\mathfrak{N}$. Let C be a closed unbounded subset of γ *all of whose members fail to be n-Mahlo.* Since $C \in \mathcal{A}$ by (✠), we can use Lemma 2.1.2 to get hold of a Σ_1-e.e.e. $(\mathfrak{M}^*, C^*)$ of $(\mathfrak{M}, C)$ with a least new ordinal γ^*. Since $(\mathfrak{M}, C)$ satisfies the Σ_0-statement

"C^* is closed and no member of C^* is an n-Mahlo cardinal",

$(\mathfrak{M}^*, C^*)$ satisfies the same sentence. But then we have a contradiction because on one hand $\gamma^* \in C^*$ since C^* is closed, thus implying that γ^* is not n-Mahlo in the eyes of $\mathfrak{M}^*$, and on the other hand γ^* is n-Mahlo in $\mathfrak{M}^*$ by our inductive hypothesis. Therefore γ must be $(n+1)$-Mahlo in $\mathfrak{N}$. □

Proof of part 2 of Theorem 2.1. We use the strategy of Schmerl and Shelah [SS] of building models with "built-in" e.e.e.'s using Mahlo cardinals of finite order. Let T be a consistent completion of $ZFC + \Phi$. We wish to describe a theory $\overline{T}$ extending T in a new language $\mathcal{L} \supseteq \{\in, \lhd\}$ with the following two key properties:

$(*)$ $ZF(\mathcal{L}) + GW \subseteq \overline{T}$.

$(**)$ Every model $\mathfrak{M}$ of $\overline{T}$ has a *conservative* e.e.e..

Note that if $\mathfrak{M}$ is a model of $\overline{T}$ and $\mathcal{A}$ is the family of $\mathfrak{M}$-definable subsets of M, then by $(*)$ $(\mathfrak{M}, \mathcal{A})$ is a model of GBC. Moreover, it is easy to see[5] that $(**)$ implies that $(\mathfrak{M}, \mathcal{A})$ satisfies "**Ord** is weakly compact". *Therefore, to prove part (2) of Theorem 2.1 it is sufficient to construct a consistent extension $\overline{T}$ of T satisfying properties $(*)$ and $(**)$ above.* This will be accomplished in Lemmas 2.1.4 and 2.1.5.

The language $\mathcal{L}$ is obtained by enriching $\{\in\}$ with a binary relation symbol $\lhd$, and two sets $\{S_n : n \in \omega\}$ and $\{U_n : n \in \omega\}$ of unary predicates.

- Let $\mathcal{L}_0 := \{\in, \lhd\}$, and $\mathcal{L}_{n+1} := \mathcal{L}_0 \cup \{S_i : i \leq n+1\} \cup \{U_i : i \leq n+1\}$.

The desired theory $\overline{T}$ is T augmented with 4 sets of axioms $\mathcal{A}_1$ through $\mathcal{A}_4$ in the extended language $\mathcal{L}$, as described below:

- $\mathcal{A}_1 := ZF(\mathcal{L}) + GW(\lhd)$.

[5]See the proof of (3) ⇒ (1) of Lemma 3.3 for more detail.

- $\mathcal{A}_2 := \{\sigma_n : n \in \omega\}$, where σ_n asserts that S_{n+1} is a satisfaction predicate over $(\mathbf{V}, \in, \lhd, S_i, U_i)_{i \leq n}$. More specifically, σ_n says that S_{n+1} contains the $\Sigma_0(\mathcal{L}_n)$-theory of $(\mathbf{V}, \in, \lhd, S_i, U_i)_{i \leq n}$, and S_{n+1} satisfies Tarski's inductive truth clauses.
- $\mathcal{A}_3 := \{\psi_n : n \in \omega\}$, where ψ_n asserts that U_{n+1} is a class of ordinal codes of unary formulas that describe a nonprincipal **Ord**-complete ultrafilter over subsets of **Ord** that are definable in

$$(\mathbf{V}, \in, \lhd, S_i, U_i)_{i \leq n} \ .$$

  Note that by $GW(\lhd)$ there is a definable bijection between the universe and the class of ordinals, and therefore, every object can be canonically coded by an ordinal. Also note that to formulate ψ_n we need to use the predicate S_{n+1} to express definability within $(\mathbf{V}, \in, S_i, U_i)_{i \leq n}$. Finally, we point out that the **Ord**-complete condition of U_{n+1} can be expressed in one sentence asserting

$$\ulcorner \tau(x, c) \in \kappa \urcorner \in U_{n+1} \Rightarrow \exists \alpha < \kappa \ulcorner \tau(x, c) = \alpha \urcorner \in U_{n+1},$$

  where τ is a definable term of $\mathcal{L}_n$, c is a parameter in $\mathbf{V}$, κ is a cardinal, and $\varphi \mapsto \ulcorner \varphi \urcorner$ is a canonical coding of formulas by ordinals.
- $\mathcal{A}_4 := \{U_n \subseteq U_{n+1} : n \in \omega\}$.

The following proposition is essentially due to Schmerl and Shelah [SS] and is our main tool in the construction of a model of $\overline{T}$. First we need a definition:

- Suppose $\mathfrak{A} = (A, \cdots)$ and $\mathfrak{B} = (B, \cdots)$ are models in common language, and $\mathfrak{A} \prec \mathfrak{B}$. $\mathfrak{A}$ is *relatively saturated* in $\mathfrak{B}$, written $\mathfrak{A} \prec_{RS} \mathfrak{B}$, if for every $X \subseteq A$ with $|X| < |A|$, every 1-type over X realized in $\mathfrak{B}$ is already realized in $\mathfrak{A}$, i.e., for every $b \in B$, there is some $a \in A$ such that

$$(\mathfrak{A}, x, a)_{x \in X} \equiv (\mathfrak{B}, x, b)_{x \in X} \ .$$

Proposition ♦. *Suppose θ is an $(n+1)$-Mahlo cardinal for some $n \in \omega$, and for some $k \in \omega$, $P_i \subseteq V_\theta$ for $i < k$. There is an n-Mahlo cardinal $\gamma < \theta$ such that $(V_\gamma, \in, P_i \cap V_\gamma)_{i<k} \prec_{RS} (V_\theta, \in, P_i)_{i<k}$.*

Proof. Let $C_1 := \{\alpha < \theta : (V_\alpha, \in, P_i \cap V_\alpha)_{i<k} \prec (V_\theta, \in, P_i)_{i<k}\}$. C_1 is a closed subset of θ by Tarski's elementary chain theorem, and it is unbounded in θ by a Skolem hull argument. Since $(V_\theta, \in) \models ZF$,

$(*)$ $\forall \alpha \in C_1 \ |V_\alpha| = \beth_\alpha = \alpha$.

Let C_2 be the subset of C_1 consisting of elements $\alpha \in C_1$ satisfying:

$$\forall \gamma < \alpha \ \forall a \in V_\theta \ \exists b \in V_\alpha \ (V_\theta, x, a)_{x \in V_\gamma} \equiv (V_\theta, x, b)_{x \in V_\gamma}.$$

Note that if $\gamma \in C_2$ is an inaccessible cardinal, then

$$(V_\gamma, \in, P_i \cap V_\gamma)_{i<k} \prec_{RS} (V_\theta, \in, P_i)_{i<k}.$$

So the proof would be complete once we verify that C_2 is a closed unbounded subset of θ. It is clear that C_2 is a closed subset of θ. To see that it is also unbounded, suppose $\alpha < \theta$. Using the inaccessibility of θ we can define a sequence $\langle \alpha_n : n \in \omega \rangle$ of elements of C_1 such that the following two conditions are satisfied for every $n \in \omega$:

(1) $\alpha < \alpha_n < \alpha_{n+1} < \theta$;

(2) Every 1-type realized in $(V_\theta, \in, P_i, x)_{i<k,\ x\in V_{\alpha_n}}$ is realized already by an element in $V_{\alpha_{n+1}}$.

Let $\delta := \bigcup_{n\in\omega} \alpha_n$. Clearly $\delta > \alpha$ and $\delta \in C_2$ because of (1), (2) and (∗). □

Lemma 2.1.4. *$\overline{T}$ has a model.*

Proof. Since $\Phi \subseteq T$, by the compactness theorem there is a model $\mathfrak{M}$ of T containing a *nonstandard* element $H \in \omega^{\mathfrak{M}}$, and some $\theta_0 \in \mathbf{Ord}^{\mathfrak{M}}$ such that

$$V_{\theta_0}^{\mathfrak{M}} \prec \mathfrak{M} \text{ and } \mathfrak{M} \models \text{“}\theta_0 \text{ is } (H+1)\text{-Mahlo”}.$$

Fix $r \in M$ such that $\mathfrak{M} \models$ “r well-orders V_{θ_0}”. Since θ is an inaccessible cardinal of $\mathfrak{M}$,

$$(V_{\theta_0}, \in, r)^{\mathfrak{M}} \models ZF(\mathcal{L}_0) + GW(\lhd).$$

Reasoning within $\mathfrak{M}$, we plan to construct a *strictly decreasing* sequence $\langle \theta_j : j < H\rangle$ of elements of $\mathbf{Ord}^{\mathfrak{M}}$, and two sequences of elements $\langle s_j : j < H\rangle$ and $\langle u_j : j < H\rangle$ of $\mathfrak{M}$ to construct the sequence of models $\langle \mathfrak{A}_j : j < H\rangle$ such that $\forall j < H\ P(j)$ holds in $\mathfrak{M}$, where

- $P(j) :=$ “For all $\mathcal{L}_j$-sentences $\varphi \in \overline{T}$, $\mathfrak{A}_j \models \varphi$”, and
- $\mathfrak{A}_j := (V_{\theta_j}, \in, r_j, s_i, u_i)_{i\leq j}$, where u_i and s_i respectively interpreteU_i and S_i, and $r_j := r \cap V_{\theta_j}$ interprets $\lhd$.

Note that $\overline{T}$ is a recursive theory and therefore we can meaningfully speak about $\overline{T}$ in $\mathfrak{M}$ (with the proviso that $\left(\overline{T}\right)^{\mathfrak{M}}$ contains formulas of infinite length and properly extends $\overline{T}$). The following recursive construction of length H should be understood to take place within $\mathfrak{M}$.

To begin with, let $s_0 = u_0 = \emptyset$. Note that $P(0)$ holds. By Proposition ⧫ we can choose an H-Mahlo cardinal $\theta_1 < \theta_0$, such that:

(1) $(V_{\theta_1}, \in, r_1) \prec (V_{\theta_0}, \in, r)$.

(note: for this step, we do not require relative saturation). Let $s_1 \subseteq \theta_1$ code the satisfaction predicate for $(V_{\theta_1}, \in, r_1)$, and let $\mathcal{U}_1$ be the ultrafilter on the parametrically $(V_{\theta_1}, \in, r_1)$-definable subsets of θ_1 that is generated by $\alpha_0 := \theta_1 \in \theta_0$, i.e., if $Y \subseteq \theta_1$ is definable in $(V_{\theta_1}, \in, r_1)$ by a formula $\varphi(x)$, then

$$Y \in \mathcal{U}_1 \text{ iff } (V_{\theta_0}, \in, r) \models \varphi(\alpha_0).$$

Let $u_1 \subseteq \theta_1$ code $\mathcal{U}_1$. Note that $P(1)$ holds. The next step of the construction is more subtle. By Proposition ⧫, there is some $(H-1)$-Mahlo cardinal $\theta_2 < \theta_1$ such that:

(2) $(V_{\theta_2}, \in, r_2, s_1 \cap \theta_2, u_1 \cap \theta_2) \prec_{RS} \mathfrak{A}_1 := (V_{\theta_1}, \in, r_1, s_1, u_1)$.

Let $s_2 \subseteq \theta_2$ code the satisfaction predicate for $(V_{\theta_2}, \in, r_2, s_1 \cap \theta_2, u_1 \cap \theta_2)$. We wish to extend the ultrafilter coded by $u_1 \cap \theta_2$ to an ultrafilter $\mathcal{U}_2$ on the family of all subsets Y of θ_2 that are definable in the model $(V_{\theta_2}, \in, r_2, s_1 \cap \theta_2, u_1 \cap \theta_2)$ such that $P(2)$ holds. By (2) there is some $\alpha_1 \in \theta_1$ such that α_0 and α_1 have the same $\mathcal{L}_1$-type in $\mathfrak{A}_1$ over V_{θ_2}. Therefore, we can use α_1 to define the desired $\mathcal{U}_2$ extending $\mathcal{U} \cap \theta_{j+1}$ via

$$Y \in \mathcal{U}_2 \text{ iff } (V_{\theta_1}, \in, r_1, s_1, u_1) \models \varphi(\alpha_1),$$

where φ defines Y in $(V_{\theta_2}, \in, r_2, s_1 \cap \theta_2, u_1 \cap \theta_2)$. It is clear that if $u_2 \subseteq \theta_2$ codes $\mathcal{U}_2$, then $P(2)$ holds. Since θ_0 was chosen to be $(H+1)$-Mahlo, thanks to Proposition ⧫

this process can be continued a total of H steps to produce sequences $\langle\theta_j : j < H\rangle$, $\langle\alpha_j : j < H\rangle$, $\langle s_j : j < H\rangle$ and $\langle u_j : j < H\rangle$ such that for all $j < H$ the following hold in $\mathfrak{M}$:

(a) θ_{j+1} is $(H-j)$-Mahlo and $\alpha_{j+1} \in \theta_{j+1}$;
(b) α_j and α_{j+1} have the same $\mathcal{L}_{j+1}$-type in $\mathfrak{A}_{j+1}$ over $V_{\theta_{j+2}}$;
(c) $u_{j+1} \subseteq \theta_{j+1}$ codes the ultrafilter generated by α_j;
(d) $s_{j+1} \subseteq \theta_{j+1}$ codes the satisfaction predicate for $(V_{\theta_{j+1}}, \in, r_{j+1}, s_i, u_i)_{i\leq j+1}$.

This makes it evident that $\mathfrak{M} \models \forall j < H\ P(j)$. The desired model of $\overline{T}$ is

$$\left((V_{\theta_K}, \in)^{\mathfrak{M}}, r_K, s_i, u_i\right)_{i<\omega},$$

where $K < H$ is any *nonstandard* element of $\omega^{\mathfrak{M}}$. $\square$

Lemma 2.1.5. *Every model $\mathfrak{M}$ of $\overline{T}$ has a conservative e.e.e. .*

Proof. Suppose $\mathfrak{M} = (M, E, \cdots)$ is a model of $\overline{T}$. Let

$$U := \bigcup_{n\in\omega} (U_n)_E\,.$$

By design, U codes an $\mathbf{Ord}^{\mathfrak{M}}$-complete nonprincipal ultrafilter $\mathcal{U}$ over the Boolean algebra of subsets of $\mathbf{Ord}^{\mathfrak{M}}$ that are parametrically definable in $\mathfrak{M}$. Let $\mathfrak{N}$ be the definable ultrapower[6] of $\mathfrak{M}$ modulo $\mathcal{U}$. Usual arguments show that $\mathfrak{N}$ is an e.e.e. of $\mathfrak{M}$. Moreover $\mathfrak{N}$ is a conservative extension of $\mathfrak{M}$ since any $c \in N$ is of the form $[\tau]_{\mathcal{U}}$ for some $\mathcal{L}_n$-term τ, and therefore

$$c_F \cap M = \{m \in M : \mathfrak{M} \models \text{“}\{\alpha \in \mathbf{Ord} : m \in \tau(\alpha)\} \in U_n\text{”}\}. \qquad \square$$

A Canonical Partition Relation

We close this section by discussing an important combinatorial principle that will be used in the proof of Theorem A.

- For a natural number $n \geq 1$, let $\mathbf{Ord} \rightarrow (\mathbf{Ord})^n$ be the statement in class theory asserting that for every $f : [\mathbf{Ord}]^n \rightarrow \{0,1\}$ there is a proper class $H \subseteq \mathbf{Ord}$ such that f is constant on $[H]^n$.
- Given $f : [\mathbf{Ord}]^n \rightarrow \mathbf{Ord}$, and $H \subseteq \mathbf{Ord}$, H is *f-canonical* if there is some $S \subseteq \{1, \cdots, n\}$ such that for all sequences $\alpha_1 < \cdots < \alpha_n$, and $\beta_1 < \cdots < \beta_n$ of elements of H,
$$f(\alpha_1, \cdots, \alpha_n) = f(\beta_1, \cdots, \beta_n) \quad \Leftrightarrow \quad \forall i \in S\ (\alpha_i = \beta_i).$$
Note that if $S = \emptyset$, then f is constant on $[H]^n$, and if $S = \{1, \cdots, n\}$, then f is injective on $[H]^n$.
- $\mathbf{Ord} \rightarrow *(\mathbf{Ord})^n$ is the statement in class theory asserting that for every $f : [\mathbf{Ord}]^n \rightarrow \mathbf{Ord}$ there is some proper class $H \subseteq \mathbf{Ord}$ such that H is f-canonical.

It is a theorem of ZFC [Bau, Theorem 2] that if κ is a weakly compact cardinal then $\kappa \rightarrow *(\kappa)^n$ holds for all $n < \omega$. This is based on Baumgartner's observation that the proof of the Erdös-Rado canonical partition theorem [ER] for $\aleph_0 \rightarrow *(\aleph_0)^n$ shows that $\kappa \rightarrow *(\kappa)^n$ follows from $\kappa \rightarrow (\kappa)^{2n}$.

[6]See the proof of part of Lemma 3.3 for more detail on the construction of such ultrapowers.

Theorem 2.2.
1. $\forall n \in \omega$, $GBC +$ "**Ord** is weakly compact" $\vdash$ **Ord** $\rightarrow (\mathbf{Ord})^n$.
2. $\forall n \in \omega$, $GBC +$ "**Ord** is weakly compact" $\vdash$ **Ord** $\rightarrow *(\mathbf{Ord})^n$.

Proof. The usual ramification tree proof from ZFC establishing the partition properties of weakly compact cardinals (as in [Kan, Theorem 7.8]) works in the GBC context to prove (1) as well, by an *external* induction on n. (2) is a corollary of (1), and the fact that the aforementioned Erdös-Rado ZFC proof of $\kappa \rightarrow *(\kappa)^n$ from $\kappa \rightarrow (\kappa)^{2n}$ can be conveniently implemented in the GBC context to establish

$$GBC + \mathbf{Ord} \rightarrow (\mathbf{Ord})^{2n} \vdash \mathbf{Ord} \rightarrow *(\mathbf{Ord})^n. \qquad \square$$

Remark 2.2.1. If GBC is replaced by KMC (Kelley-Morse theory of classes with global choice), then in both parts of Theorem 2.2 "$\forall n \in \omega$" can be moved to the right hand side of the provability symbol $\vdash$.

3. AUTOMORPHISMS FROM MAHLO CARDINALS

The main result of this section is Theorem A, which is the analogue of Theorem 1.1 for $ZFC + \Phi$. In the next section we shall prove a "reversal" of Theorem A by showing that over the weak fragment $EST(\{\in, \lhd\}) + GW^*$ the assumption that T contains $ZFC + \Phi$ is indeed necessary.

Theorem A. *Suppose T is a consistent completion of $ZFC+ \Phi$. There is a model $\mathfrak{M}$ of $T + ZF(\lhd) + GW$ such that $\mathfrak{M}$ has a proper e.e.e. $\mathfrak{N}$ that possesses an automorphism whose fixed point set is M.*

The proof of Theorem A is presented at the end of the present section, once the machinery of generic ultrafilters and iterated ultrapowers are put into place. However, we can describe the high-level strategy of the proof here: $\mathfrak{M}$ is the $\{\in, \lhd\}$-reduct of the model of GBC + "**Ord** is weakly compact" whose existence is certified by part (2) of Theorem 2.1; $\mathfrak{N}$ is the $\mathbb{Z}$-iterated ultrapower of $\mathfrak{M}$ modulo a "generic ultrafilter" (where $\mathbb{Z}$ is the linearly ordered set of integers); and the desired automorphism of $\mathfrak{N}$ is induced by the automorphism $n \mapsto n+1$ of $\mathbb{Z}$.

For a countable model $(\mathfrak{M}, \mathcal{A})$ of GBC, let $\mathbb{B}$ be the Boolean algebra

$$\{S \in \mathcal{A} : S \subseteq \mathbf{Ord}^{\mathfrak{M}}\}.$$

Our first goal is to construct ultrafilters $\mathcal{U}$ over $\mathbb{B}$ with certain combinatorial properties that yield desirable structural properties of $\mathcal{U}$-based ultrapowers. We shall employ the conceptual framework of *forcing* in order to efficiently present the necessary bookkeeping arguments. Our notion of forcing is the poset $\mathbb{P}$, where

$$\mathbb{P} := \{S \in \mathbb{B} : S \text{ is unbounded in } \mathbf{Ord}^{\mathfrak{M}}\},$$

ordered under inclusion.

- A subset $\mathcal{D}$ of $\mathbb{P}$ is *dense* if for every $X \in \mathbb{P}$ there is some $Y \in \mathcal{D}$ with $Y \subseteq X$.
- $\mathcal{U} \subseteq \mathbb{P}$ is a filter if it satisfies (1) $\emptyset \notin U$; (2) $\mathcal{U}$ is closed under intersections; and (3) $\mathcal{U}$ is upward closed.
- A filter $\mathcal{U} \subseteq \mathbb{P}$ is $\mathbb{P}$-*generic over* $(\mathfrak{M}, \mathcal{A})$ if $\mathcal{U} \cap \mathcal{D} \neq \emptyset$ whenever $\mathcal{D}$ is a dense subset of $\mathbb{P}$ that is parametrically definable in $(\mathfrak{M}, \mathcal{A})$.

- A filter $\mathcal{U} \subseteq \mathbb{P}$ is $(\mathfrak{M}, \mathcal{A})$-*complete* if for every $f : \mathbf{Ord}^{\mathfrak{M}} \to \kappa$, where $\kappa \in \mathbf{Ord}^{\mathfrak{M}}$ and $f \in \mathcal{A}$, there is some $H \in \mathcal{U}$ such that f is constant on $\mathcal{U}$.

Note that if a filter $\mathcal{U} \subseteq \mathbb{P}$ is $(\mathfrak{M}, \mathcal{A})$-complete, then $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{B}$ since for each $Y \in \mathbb{B}$, the characteristic function of Y is constant on some member of $\mathcal{U}$. We therefore refer to $(\mathfrak{M}, \mathcal{A})$-complete filters as *ultrafilters.*

- Let Γ be a canonical bijection between $\mathbf{Ord}^{\mathfrak{M}} \times \mathbf{Ord}^{\mathfrak{M}}$ and $\mathbf{Ord}^{\mathfrak{M}}$. Every $g : \mathbf{Ord}^{\mathfrak{M}} \to \{0,1\}$ codes a sequence $\left\langle S^g_\alpha : \alpha \in \mathbf{Ord}^{\mathfrak{M}} \right\rangle$ of subsets of $\mathbf{Ord}^{\mathfrak{M}}$, where $S^g_\alpha := \{\beta \in \mathbf{Ord}^{\mathfrak{M}} : g(\Gamma(\alpha, \beta)) = 1\}$.
- A filter $\mathcal{U} \subseteq \mathbb{P}$ is $(\mathfrak{M}, \mathcal{A})$-*iterable*[7] if $\mathcal{U}$ is $(\mathfrak{M}, \mathcal{A})$-complete, and for every $g \in \mathcal{A}$ and $g : \mathbf{Ord}^{\mathfrak{M}} \to \{0,1\}$, $\{\alpha \in \mathbf{Ord} : S^g_\alpha \in \mathcal{U}\} \in \mathcal{A}$.
- A filter $\mathcal{U} \subseteq \mathbb{P}$ is $(\mathfrak{M}, \mathcal{A})$-*canonically Ramsey* if for every
$$f : [\mathbf{Ord}^{\mathfrak{M}}]^n \to \mathbf{Ord}^{\mathfrak{M}},$$
where n is a standard natural number, and $f \in \mathcal{A}$, f is canonical on some $H \in \mathcal{U}$, i.e., there is some $S \subseteq \{1, \cdots, n\}$ such that for all sequences $\alpha_1 < \cdots < \alpha_n$, and $\beta_1 < \cdots < \beta_n$ of elements of H,
$$f(\alpha_1, \cdots, \alpha_n) = f(\beta_1, \cdots, \beta_n) \Leftrightarrow \forall i \in S \ (\alpha_i = \beta_i).$$

The usual proof establishing the existence of filters meeting countably many dense sets shows:

Proposition 3.1. *If $(\mathfrak{M}, \mathcal{A})$ is a countable model of GBC, then there is a generic filter $\mathcal{U}$ over $(\mathfrak{M}, \mathcal{A})$.*

The following result reveals the key properties of generic ultrafilters.

Theorem 3.2. *If $(\mathfrak{M}, \mathcal{A})$ is a model of GBC + "**Ord** is weakly compact" and $\mathcal{U}$ is $(\mathfrak{M}, \mathcal{A})$-generic, then*

1. *$\mathcal{U}$ is $(\mathfrak{M}, \mathcal{A})$-complete;*
2. *$\mathcal{U}$ is $(\mathfrak{M}, \mathcal{A})$-iterable;*
3. *$\mathcal{U}$ is $(\mathfrak{M}, \mathcal{A})$-canonically Ramsey.*

Proof. (1): Given $f \in \mathcal{A}$ with $f : \mathbf{Ord}^{\mathfrak{M}} \to \kappa$, and $\kappa \in \mathbf{Ord}^{\mathfrak{M}}$, let
$$\mathcal{D}^f_1 := \{Y \in \mathbb{P} : f \restriction Y \text{ is constant}\}.$$
$\mathcal{D}^f_1$ is dense since $\mathbf{Ord}^{\mathfrak{M}}$ behaves like a regular cardinal in models $(\mathfrak{M}, \mathcal{A})$ of GBC.

(2): For X and Y in $\mathbb{B}$, let us write $X \subseteq_* Y$ (read "X is almost contained in Y") if $X \backslash Y$ is bounded in $\mathbf{Ord}^{\mathfrak{M}}$ (equivalently: $X \backslash Y$ is a set in $\mathfrak{M}$). Also, let "X decides Y" abbreviate
$$\text{"}X \subseteq_* Y \text{ or } X \subseteq_* \mathbf{Ord}^{\mathfrak{M}} \backslash Y\text{"}.$$
For each $g : \mathbf{Ord}^{\mathfrak{M}} \to \{0,1\}$, with $g \in \mathcal{A}$, let
$$\mathcal{D}^g_2 := \{Y \in \mathbb{P} : \forall \alpha \in \mathbf{Ord}^{\mathfrak{M}} \ Y \text{ decides } S^g_\alpha\}.$$

[7] This notion of iterability is related to, but different from, its namesake in the theory of large cardinals used in [Kan, p.249]. In the present context iterability provides a useful combinatorial condition for the ultrapower construction to be iterated, using finite supports, along any linearly ordered set. Our iterable ultrafilters are precisely the analogues of Kunen's "M-ultrafilters" (as in [Ku], or [Jec]) for models of GBC.

We observe that if every $\mathcal{D}_2^g$ is dense, then (2) holds. To show the density of $\mathcal{D}_2^g$ suppose $X \in \mathbb{P}$. We first claim that there is an $\mathcal{A}$-coded sequence $F = \left\langle F_\alpha : \alpha \in \mathbf{Ord}^{\mathfrak{M}} \right\rangle$ satisfying the following two properties:

(*) $\forall \alpha \in \mathbf{Ord}^{\mathfrak{M}}\ F_\alpha = S_\alpha^g \cap X$ or $F_\alpha = X \backslash S_\alpha^g$;

(**) $\forall \alpha \in \mathbf{Ord}^{\mathfrak{M}} \bigcap\limits_{\delta<\alpha} F_\delta$ is unbounded in X.

Argue within $(\mathfrak{M}, \mathcal{A})$. For each $s : \alpha \to \{0,1\}$, define $\langle F_\delta^s : \delta < \alpha \rangle$ by:

$$F_\delta^s := \begin{cases} S_\delta^g \cap X, & \text{if } s(\delta) = 1; \\ X \backslash S_\delta^g, & \text{if } s(\delta) = 0. \end{cases}$$

Consider the subtree τ of $2^{<\mathbf{Ord}}$ consisting of function $s : \alpha \to \{0,1\}$ such that

$$\bigcap_{\delta<\alpha} F_\delta^s \text{ is unbounded in } X.$$

It is easy to see that τ has nodes of every rank $\alpha \in \mathbf{Ord}$, because each level of τ gives rise to a partition of X into $2^{|\alpha|}$ pieces, and so by Power Set and Replacement, one of the pieces must be a proper class since X itself is a proper class. By weak compactness of $\mathbf{Ord}$, τ has a branch in $\mathcal{A}$, which is the desired sequence $\langle F_\alpha : \alpha \in \mathbf{Ord} \rangle$. We can now define a proper class $Y = \{y_\alpha : \alpha < \mathbf{Ord}\}$ by transfinite induction within $(\mathfrak{M}, \mathcal{A})$ such that Y is almost contained in every F_α as follows:

- y_0 is the first element of F_0;
- For $\alpha > 0$, y_α is the least member of $\bigcap\limits_{\delta\leq\alpha} F_\delta$ above $\{y_\delta : \delta < \alpha\}$.

It is clear that Y decides each S_α^g. Therefore $\mathcal{D}_2^g$ is dense.

(3): Suppose $f : [\mathbf{Ord}^{\mathfrak{M}}]^n \to \mathbf{Ord}^{\mathfrak{M}}$, where n is a standard natural number, and $f \in \mathcal{A}$. Let

$$\mathcal{D}_3^f := \{Y \in \mathbb{P} : f \text{ is canonical on } Y\}.$$

By Theorem 2.2, $\mathcal{D}_3^f$ is dense. □

Remark 3.2.1. In sharp contrast to the usual scenario in the theory of large cardinals, *a $\mathbb{P}$-generic filter is never normal.* This is because if $f \in \mathcal{A}$ and for every $\alpha \in \mathbf{Ord}^{\mathfrak{M}}\ f^{-1}\{\beta \in \mathbf{Ord}^{\mathfrak{M}} : \beta > \alpha\} \in \mathbb{P}$, then the following set $\mathcal{D}_4^f$ forms a dense subset of $\mathbb{P}$:

$$\mathcal{D}_4^f := \{Y \in \mathbb{P} : \exists g : Y \to^{1-1} Y \text{ such that } g \in \mathcal{A} \text{ and } g < f \text{ on } Y\}.$$

We are now in a position to examine ultrapowers and their iterations.

Lemma 3.3. *The following are equivalent for a countable model $(\mathfrak{M}, \mathcal{A})$ of GBC :*

1. $(\mathfrak{M}, \mathcal{A}) \models$ "**Ord** *is weakly compact*".
2. *There is a nonprincipal $(\mathfrak{M}, \mathcal{A})$-iterable ultrafilter $\mathcal{U}$.*
3. *$(\mathfrak{M}, S)_{S\in\mathcal{A}}$ has a proper conservative e.e.e. $(\mathfrak{N}, S^*)_{S\in\mathcal{A}}$.*

Proof. (1) ⇒ (2) follows from Proposition 3.1 and part (2) of Theorem 3.2. For (2) ⇒ (3), let $(\mathfrak{N}, S^*)_{S\in\mathcal{A}}$ be the ultrapower of $(\mathfrak{M}, S)_{S\in\mathcal{A}}$ modulo $\mathcal{U}$. More precisely, let $\mathcal{F}$ be the family of functions f from $\mathbf{Ord}^{\mathfrak{M}}$ into M such that f is canonically coded by some element of $\mathcal{A}$. For $f \in \mathcal{F}$, let $[f]_{\mathcal{U}}$ be the $\mathcal{U}$-equivalence class of f consisting of members of $\mathcal{F}$ that agree with f on a member of $\mathcal{U}$. Define $\mathfrak{N} := (N, F)$, and S^* for $S \in \mathcal{A}$ as follows

$$N := \{[f]_{\mathcal{U}} : f \in \mathcal{F}\};$$

$$[f]\ F\ [g] \text{ iff } \{\alpha \in \mathbf{Ord}^{\mathfrak{M}} : f(\alpha)\ E\ g(\alpha)\} \in \mathcal{U}, \text{ where } E = \ \in^{\mathfrak{M}};$$

$$[f]_{\mathcal{U}} \in S^* \text{ iff } \{\alpha \in \mathbf{Ord}^{\mathfrak{M}} : f(\alpha) \in S\} \in \mathcal{U}.$$

Thanks to the presence of a global well-ordering in $\mathcal{A}$, the Łoś Theorem for ultrapowers holds in this context. Consequently, if $\mathcal{U}$ is a non-principal ultrafilter, then $\mathfrak{N}$ is a proper elementary extension of $\mathfrak{M}$ (with the obvious identification of the $\mathcal{U}$-equivalence classes of constant maps with elements of $\mathfrak{M}$). The $(\mathfrak{M},\mathcal{A})$-completeness property of $\mathcal{U}$, coupled with the existence of a global well-ordering in $\mathcal{A}$, assures us that $\mathfrak{N}$ is an end extension of $\mathfrak{N}$. To verify the conservativity clause one invokes the conservativity clause to show that $\mathcal{A} = \{c_F \cap M : c \in N\}$[8].

To establish (3) $\Rightarrow$ (1), suppose τ is an $\mathbf{Ord}$-tree coded in $\mathcal{A}$. We may assume without loss of generality that $\tau = (\mathbf{Ord}^{\mathfrak{M}}, <_\tau)$, where the tree ordering $<_\tau$ satisfies

$(*)$ $(\mathfrak{M},\tau) \vDash \forall \alpha,\beta \in \mathbf{Ord}\ (\alpha <_\tau \beta \rightarrow \alpha \in \beta)$.

Therefore, τ is end extended by τ^* (in the sense that if $\alpha <_{\tau^*} \beta \in \mathbf{Ord}^{\mathfrak{M}}$ then $\alpha \in \mathbf{Ord}^{\mathfrak{M}}$). Now let $\delta \in \mathbf{Ord}^{\mathfrak{N}} \backslash \mathbf{Ord}^{\mathfrak{M}}$, and define

$$B := \{\alpha \in \mathbf{Ord}^{\mathfrak{M}} : \alpha <_{\tau^*} \delta\}.$$

Since $\tau \prec_e \tau^*$ and $(*)$ holds, B forms a branch of τ, and by conservativity, $B \in \mathcal{A}$.

Note that the equivalence of (2) and (3), as well as (3) $\Rightarrow$ (1) do not require the countability condition on $(\mathfrak{M},\mathcal{A})$. $\square$

For an $(\mathfrak{M},\mathcal{A})$-iterable ultrafilter $\mathcal{U}$, the fact that the $\mathcal{U}$-based ultrapower does not introduce new subsets of $\mathbf{Ord}^{\mathfrak{M}}$ allows one to iterate the ultrapower formation any finite number of times to obtain the finite iteration $Ult_{\mathcal{U},n}(\mathfrak{M},S)_{S\in\mathcal{A}}$. Indeed, the finite iteration can also be obtained in *one step* by defining an ultrafilter $\mathcal{U}^n$ on $\left(\mathbf{Ord}^{\mathfrak{M}}\right)^n$. To do so, suppose $X \subseteq \left(\mathbf{Ord}^{\mathfrak{M}}\right)^n$, where X is coded in $\mathcal{A}$. By definition, $X \in \mathcal{U}^{n+1}$ iff

$$\{\alpha_1 : \{(\alpha_2,\cdots,\alpha_{n+1}) : (\alpha_1,\alpha_2,\cdots,\alpha_{n+1}) \in X\} \in \mathcal{U}^n\} \in \mathcal{U}.$$

(the iterability condition ensures that $\mathcal{U}^{n+1}$ is well-defined). Moreover, the process of ultrapower formation modulo $\mathcal{U}$ can be iterated *along any linear order* $\mathbb{L}$ to yield the iterated ultrapower $Ult_{\mathcal{U},\mathbb{L}}(\mathfrak{M},S)_{S\in\mathcal{A}}$. There are two equivalent ways of describing the isomorphism type of $Ult_{\mathcal{U},\mathbb{L}}(\mathfrak{M},S)_{S\in\mathcal{A}}$:

1. *The model theoretic approach*: This is essentially Gaifman's iteration method in his treatment of models of arithmetic [G]. Using an iterable ultrafilter $\mathcal{U}$ one first defines, for each positive natural number n, a complete n-type Γ_n over the model $(\mathfrak{M},S)_{S\in\mathcal{A}}$ by defining $\Gamma_n(x_1,\cdots,x_n)$ as the set of formulas $\varphi(x_1,\cdots,x_n)$ such that

$$\{(\alpha_1,\cdots,\alpha_n) : (\mathfrak{M},S)_{S\in\mathcal{A}} \vDash \varphi(\alpha_1,\cdots,\alpha_n)\} \in \mathcal{U}^n.$$

Here φ is a formula in the language $\mathcal{L}^* = \{\in\} \cup \{S : S \in \mathcal{A}\}$ (since for each $m \in M$, $\{m\} \in \mathcal{A}$, for all intents and purposes $\mathcal{L}$ has constant symbols for elements of M as well). Now augment the language $\mathcal{L}^*$ with a set of new constant symbols $\{c_l : l \in \mathbb{L}\}$, and define $\Psi_{\mathbb{L}} := \{c_{l_1} \in c_{l_2} : l_1 <_{\mathbb{L}} l_2\}$ and

$$T_{\mathcal{U},\mathbb{L}} := \Psi_{\mathbf{L}} \cup \{\varphi(c_{l_1},\cdots,c_{l_1}) : \varphi(x_1,\cdots,x_n) \in \Gamma_n(x_1,\cdots,x_n)\}.$$

[8]See [Kan, Lemma 19.1(c)] for a similar proof.

$T_{\mathcal{U},\mathbb{L}}$ turns out to be a *complete Skolemized theory.* Therefore $Ult_{\mathcal{U},\mathbb{L}}(\mathfrak{M},S)_{S\in\mathcal{A}}$ can be meaningfully defined as the *prime model* of $T_{\mathcal{U},\mathbb{L}}$.

2. *The algebraic approach*: This is how iterated ultrapowers are handled in set theory à la Kunen [Ku]. Let $\mathbb{I} = [\mathbb{L}]^{<\omega}$ = the family of finite subsets of $\mathbb{L}$. One builds a category $\mathcal{C}$ whose *objects* are models of the form $\mathfrak{N}_S$, for $S \in \mathbb{I}$, and whose *morphisms* are maps

$$\pi_{S_1,S_2} : \mathfrak{N}_{S_1} \to \mathfrak{N}_{S_2}, \text{with } S_1 \subseteq S_2 \in \mathbb{I},$$

such that the following properties hold:

- $\mathfrak{N}_\emptyset = \mathfrak{M}$, and for $S \in \mathbb{I}$ with $|S| = n > 1$, $\mathfrak{N}_S = \mathfrak{M}^{(\mathbf{Ord}^{\mathfrak{M}})^n}/\mathcal{U}^n$;
- π_{S_1,S_2} is an elementary embedding, whenever $S_1 \subseteq S_2 \in \mathbb{I}$;
- $\pi_{S_1,S_3} = \pi_{S_2,S_3} \circ \pi_{S_1,S_2}$, whenever $S_1 \subseteq S_2 \subseteq S_3 \in \mathbb{I}$.

In this approach, $Ult_{\mathcal{U},\mathbb{L}}(\mathfrak{M},S)_{S\in\mathcal{A}}$ is defined as the *direct limit* of the category $\mathcal{C}$. One can then use the usual arguments from the theory of iterated ultrapowers (as in [Jec], or [Kan]) to establish the following result.

Theorem 3.4. *Suppose* $(\mathfrak{N},S^*)_{S\in\mathcal{A}} := Ult_{\mathcal{U},\mathbb{L}}(\mathfrak{M},S)_{S\in\mathcal{A}}$, *and* $\bar{l} := c_l^{\mathfrak{N}}$, *where* $(\mathfrak{M},\mathcal{A}) \models GBC$, *and* $\mathcal{U}$ *is a nonprincipal* $(\mathfrak{M},\mathcal{A})$*-iterable ultrafilter.*

1. *Elements of* $\mathfrak{N}$ *are of the form* $f^*(\overline{l_1},\cdots,\overline{l_n})$, *where* $f \in \mathcal{A}$, *and* $l_1 < \cdots < l_n$;
2. *for every* $\mathcal{L}^*$*-formula* $\varphi(x_1,\cdots,x_n)$,

$$\mathfrak{N} \models \varphi(\overline{l_1},\cdots,\overline{l_n}) \textit{ iff } \{(\alpha_1,\cdots,\alpha_n) \in \left(\mathbf{Ord}^{\mathfrak{M}}\right)^n : \mathfrak{M} \models \varphi(\alpha_1,\cdots,\alpha_n)\} \in \mathcal{U}^n;$$

3. $\{\bar{l} : l \in \mathbb{L}\}$ *is a set of order indiscernibles in* $(\mathfrak{N},S^*)_{S\in\mathcal{A}}$ *of order type* $\mathbb{L}$*;*
4. *every automorphism* h *of* $\mathbb{L}$ *induces an automorphism* j_h *of* $(\mathfrak{N},S^*)_{S\in\mathcal{A}}$ *defined by*

$$j_h(f^*(\overline{l_1},\cdots,\overline{l_n})) = f^*(\overline{h(l_1)},\cdots,\overline{h(l_n)}).$$

Theorem 3.5. *Suppose* $(\mathfrak{M},\mathcal{A}) \models GBC +$ "**Ord** *is weakly compact", and let* h *be an automorphism of a linearly ordered set* $\mathbb{L}$ *with no fixed points. If* $\mathcal{U}$ *is* $(\mathfrak{M},\mathcal{A})$*-generic, then the fixed point set of the automorphism* j_h *of* $(\mathfrak{N},S^*)_{S\in\mathcal{A}}$ $:= Ult_{\mathcal{U},\mathbb{L}}(\mathfrak{M},S)_{S\in\mathcal{A}}$ *is precisely* M.

Proof. Note that by part (2) of Theorem 3.2, $\mathcal{U}$ is $(\mathfrak{M},\mathcal{A})$-iterable and therefore Theorem 3.4 applies. Clearly j_h fixes each $a \in M$ since the constant map $f_a(x) = a$ is in $\mathcal{A}$. To see that j_h fixes no member of $N\backslash M$, suppose that for some $f^*(\overline{l_1},\cdots,\overline{l_n}) \in N$,

(1) $f^*(\overline{h(l_1)},\cdots,\overline{h(l_n)}) = f^*(\overline{l_1},\cdots,\overline{l_n})$.

Since $f \in \mathcal{A}$, by Theorem 3.2 there is some $H \in \mathcal{U}$, and some $S \subseteq \{1,\cdots,n\}$ such that for all sequences $\alpha_1 < \cdots < \alpha_n$ and $\beta_1 < \cdots < \beta_n$ of elements of H,

(2) $f(\alpha_1,\cdots,\alpha_n) = f(\beta_1,\cdots,\beta_n) \Leftrightarrow \forall i \in S\ (\alpha_i = \beta_i)$.

Moreover, since $H^n \in \mathcal{U}^n$,

(3) $(\overline{l_1},\cdots,\overline{l_n}) \in (H^*)^n$, whenever $l_1 <_{\mathbb{L}} \cdots <_{\mathbb{L}} l_n$.

(1), (2), and (3) together imply that $S = \emptyset$. So f must be *constant* on H, hence $f^*(\overline{l_1},\cdots,\overline{l_n}) \in M$. □

Proof of Theorem A. Start with a consistent completion T of $ZFC+\Phi$. By Theorem 2.1 there is a countable $\mathfrak{M}$ model of T that expands to a model $(\mathfrak{M},\mathcal{A})$ of $GBC+$"**Ord** is weakly compact". Use Proposition 3.1 to fix some generic ultrafilter $\mathcal{U}$ over $(\mathfrak{M},\mathcal{A})$, and let

$$(\mathfrak{N},S^*)_{S\in\mathcal{A}}:=Ult_{\mathcal{U},\mathbb{Z}}(\mathfrak{M},S)_{S\in\mathcal{A}},$$

where $\mathbb{Z}$ is the ordered set of integers. Consider the automorphism

$$n\longmapsto_h n+1$$

of $\mathbb{Z}$. By Theorem 3.5 j_h is an automorphism of $(\mathfrak{N},S^*)_{S\in\mathcal{A}}$ whose fixed point set is precisely M. □

4. MAHLO CARDINALS FROM AUTOMORPHISMS

In this section we turn the table around and establish a "reversal" of Theorem A. More specifically, in part (2) of Theorem B we show that over the weak fragment $EST+GW^*$ of $ZF+GW^*$ (discussed in the preliminaries section), $ZFC+\Phi$ is necessary for Theorem A. The proof of Theorem B relies on Theorem 2.1 and Lemmas 4.1 through 4.5.

- Through this section, $\mathcal{L}:=\{\in,\lhd\}$, and $\mathfrak{N}:=(N,F,\lhd)\models EST(\mathcal{L})+GW^*$.
- For $M\subseteq N$, $\mathfrak{M}$ is the submodel $(M,E,\blacktriangleleft)$ of $\mathfrak{N}$, where $E:=F\cap M^2$, and $\blacktriangleleft:=\lhd\cap M^2$.
- M is a $\lhd$-*cut* of N, if N properly $\lhd$-end extends M (i.e., $\forall a\in M\ \forall b\in N\backslash M$, $a\lhd b$) and $N\backslash M$ has no $\lhd$-least member.
- Suppose M is a $\lhd$-cut of N. M is a *strong* $\lhd$-*cut* of N, if for each function $f\in N$ whose domain includes M, there is some s in N, such that for all $m\in M$,

  $$f(m)\notin M \ \text{ iff } \ s\lhd f(m).$$

  This is the set theoretic analogue of the Kirby-Paris notion of strong cuts [KP].

Lemma 4.1. *Suppose M is a $\lhd$-cut of N.*

1. $\forall a\in N\backslash M\ \exists c\in N$ *such that* $\mathfrak{N}\models$ "$c=\{x:x\lhd a\}$", *and* $M\subseteq c_F$.
2. [$\Delta_0(\mathcal{L})$-*overspill*] *If $\varphi(x)$ is a $\Delta_0(\mathcal{L})$-formula such that every element of M is a solution of $\varphi(x)$, then there is a solution of $\varphi(x)$ in $N\backslash M$.*

Proof. (1) is an immediate consequence of GW^* and the assumption that M is $\lhd$-end extended by N. To verify (2) assume the contrary and fix $c\in N$ such that $M\subseteq c_F$. By $\Delta_0(\mathcal{L})$-Separation, for some $s\in N$,

$$\mathfrak{N}\models\ \text{"}s=\{x\in c:\neg\varphi(x)\}\neq\emptyset\text{"}\ .$$

Therefore s has a $\lhd$-least element by GW, which must be the $\lhd$-least member of $N\backslash M$, contradiction. □

Theorem B. *If j is an automorphism of $\mathfrak{N}$ whose fixed point set M is a $\lhd$-initial segment of N, and $\mathcal{A}:=\{a_F\cap M:a\in N\}$, then:*

1. $(\mathfrak{M},\mathcal{A})\models GBC+$"**Ord** *is weakly compact*";
2. $\mathfrak{M}\models ZFC+\Phi$.

Note that Theorem 2.1 shows that the second part of Theorem B follows from its first part. The proof of part (1) of Theorem B will be presented after establishing Lemmas 4.2 through 4.5. We begin verifying some basic closure properties of M.

- For the rest of this section j is an automorphism of $\mathfrak{N}$ whose fixed point set M is a proper $\lhd$-initial segment of N.

Lemma 4.2.
1. $\mathfrak{M} \subseteq_e \mathfrak{N}$.
2. *If* $a \in N$ *and* $a_F \subseteq M$, *then* $a \in M$.
3. *If* $a \in M$, $b \in N$, *and* $\mathfrak{N} \models b \subseteq a$, *then* $b \in M$.

Proof. (1) is an immediate consequence of coupling the definition of a $\lhd$-cut with GW^*. For (2), suppose $a \in N$ and $a_F \subseteq M$. Then for every $x \in N$,

$$\mathfrak{N} \models x \in a \Leftrightarrow \mathfrak{N} \models x \in j(a).$$

Hence by Extensionality, $j(a) = a$, so $a \in M$. Finally, (3) follows easily from (2). $\square$

Lemma 4.3. $\mathfrak{M}$ *satisfies* $EST(\mathcal{L}) + GW^*+$ Power Set.

Proof. Lemma 4.2 is our tool for verifying that $\mathfrak{M}$ inherits $EST(\mathcal{L}) + GW^*$ from $\mathfrak{N}$: Extensionality and Foundation are inherited by part (1); Pairs and Union are inherited by part (2); $\Delta_0(\mathcal{L})$-Separation is inherited by part (3). Infinity holds by coupling the assumption that $\mathfrak{M}$ forms a $\lhd$-initial segment of $\mathfrak{N}$, with the fact that $\omega^{\mathfrak{N}}$ is first order definable in $\mathfrak{N}$ (and is therefore fixed by every automorphism of $\mathfrak{N}$). GW^* is also easily verified by parts (1) and (2) of Lemma 4.2.

To verify Power Set in $\mathfrak{M}$, observe that by part (3) of Lemma 4.2, if $a \in M$ then all subsets of a (in the sense of $\mathfrak{N}$) are in M. By part (1) of Proposition 4.1, there is some $c \in N$ such that $M \subseteq c_F$, and by $\Delta_0(\mathcal{L})$-Separation in $\mathfrak{N}$ the set $d := \{x \in c : x \subseteq a\}$ exists in $\mathfrak{N}$. Since $d \in M$ by part (3) of Proposition 4.2, this shows that $\mathfrak{M}$ satisfies Power Set. $\square$

Lemma 4.4 unveils the most important feature of M. It will be used in Lemma 4.5 to uniformly translate global predicates in $(\mathfrak{M}, S)_{S \in \mathcal{A}}$ to local ones in $\mathfrak{N}$.

Lemma 4.4. *M is a strong $\lhd$-cut of N.*

Proof. Recall that by assumption M is a $\lhd$-initial segment of N. Clearly M is a $\lhd$-cut of N since if a is the $\lhd$-least element of $N \backslash M$, then by a reasoning similar to the proof of part (2) of Lemma 4.2, $j(a) = a$, which contradicts $a \notin M$. Next, suppose f is the graph of a function in N whose domain includes M. Let $g := j(f)$, and note that $f \notin M$, $g \notin M$, and $f \neq g$. Therefore:

(1) $\forall m \in M \ [f(m) = g(m) \Longleftrightarrow f(m) \in M]$.

We wish to find $s \in N$ such that for all $m \in M$, $f(m) \notin M$ iff $s \lhd f(m)$. Without loss of generality there is some $m_0 \in M$ with $f(m_0) \notin M$. Fix $c \in N$ such that $M \subseteq c_F$ and

$$\mathfrak{N} \models c \supseteq (f \cup (\cup f) \cup (\cup \cup f) \cup g \cup (\cup g) \cup (\cup \cup g)).$$

Let $m \in M$ with $m_0 \lhd m$, and consider the $\Delta_0(\mathcal{L})$-formula $\varphi(x)$ with parameters m, f and g:

$$\varphi(x, m) := \exists v \in c \ [v \lhd m \text{ and } (x = \ f(v) \neq g(v))].$$

Note that by $\Delta_0(\mathcal{L})$-Separation $\mathfrak{N}$ thinks that for every $m \in M$ with $m_0 \lhd m$, $\{x \in c : \varphi(x,m)\}$ exists and has a $\lhd$-least member $h(m)$. By (1), for every $m \in M$ with $m_0 \lhd m$, $h(m) \in N \backslash M$. Therefore,

(2) $\forall m \in M\ [m_0 \lhd m \Rightarrow \mathfrak{N} \vDash m \lhd h(m)]$.

Observe that the predicate $\psi(m)$ expressing $m \lhd h(m)$ is a $\Delta_0(\mathcal{L})$-formula, because all the quantifiers of ψ can be limited to c. So by $\Delta_0(\mathcal{L})$-overspill there is some $a \in N \backslash M$ such that $\mathfrak{N} \vDash a \lhd h(a)$. The desired s demonstrating the strength of M in N is $h(a)$ since

(3) $\forall m \in M\ [m_0 \lhd m \Rightarrow \mathfrak{N} \vDash h(a) \unlhd h(m)]$,

which, coupled with (2), implies

(4) $\forall m \in M\ [h(a) \lhd m \Longleftrightarrow f(m) \notin M]$. $\square$

Lemma 4.5. *Let $\mathcal{L}^* = \{\in\} \cup \{S : S \in \mathcal{A}\}$. For every $\mathcal{L}^*$-formula*

$$\varphi(\overrightarrow{x}, \overrightarrow{S}),$$

with free variables $\overrightarrow{x}$ and parameters $\overrightarrow{S}$ from $\mathcal{A}$, there is some $\Delta_0(\mathcal{L})$-formula

$$\theta_\varphi(\overrightarrow{x}, \overrightarrow{b}),$$

where $\overrightarrow{b}$ is a sequence of parameters from N, such that for all sequences $\overrightarrow{a}$ of elements of M of the same length as $\overrightarrow{x}$,

$$(\mathfrak{M}, S)_{S \in \mathcal{A}} \vDash \varphi(\overrightarrow{a}, \overrightarrow{S}) \quad \textit{iff} \quad \mathfrak{N} \vDash \theta_\varphi(\overrightarrow{a}, \overrightarrow{b}).$$

Proof. Fix an element $d \in N$ with $M \subseteq d_F$. Since EST does not guarantee the existence of Cartesian products, we first wish to show that for each standard natural number n there is some element $c_n \in N$ such that $M \subseteq (c_n)_F$ and the n-fold Cartesian product $(c_n)^n$ exists in N. By Lemma 4.3, Power Set holds in $\mathfrak{M}$, so for every standard natural number n and every $x \in M$, $\mathfrak{M}$ thinks that the Cartesian product x^n exists. Given a *fixed* standard natural number n, consider the $\Delta_0(\mathcal{L})$-formula $\psi_n(x)$:

$$\psi_n(\mathbf{x}) := \exists y \in d\ \exists z \in d\ (\text{“}y = \{t \in d : t \lhd \mathbf{x}\}\text{” and } z = y^n).$$

Since every element of M is a solution to ψ_n, by $\Delta_0(\mathcal{L})$-overspill there is some solution a of ψ_n in $N \backslash M$. The desired c_n is $\{t \in d : t \lhd a\}$.

We construct θ_φ by recursion on the complexity of φ as follows:

- If φ is an atomic formula of the form $S_i(v)$, where v is a term, then choose $b \in N$ such that $b_F \cap M = S_i$, and define $\theta_\varphi := (v \in b)$. For other atomic formulas φ, $\theta_\varphi := \varphi$.
- If $\varphi = \neg\delta$, then $\theta_\varphi := \neg\theta_\delta$;
- If $\varphi = \delta_1 \vee \delta_2$, then $\theta_\varphi := \theta_{\delta_1} \vee \theta_{\delta_2}$;
- If $\varphi = \exists v \delta(v, \overrightarrow{x})$ and the length of $\overrightarrow{x}$ is n, then consider the function $f(\overrightarrow{x})$ defined in $\mathfrak{N}$ on $(c_{n+1})^n$ by

  $$f(\overrightarrow{x}) := \text{ the } \lhd\text{-least } v \in c_{n+1} \text{ such that } \theta_\delta(v, \overrightarrow{x}),$$

  if $\exists v \in c_{n+1}\ \theta_\delta(v, \overrightarrow{x})$; and 0 otherwise. Note that the graph of f is defined by a $\Delta_0(\mathcal{L})$-formula within $\mathfrak{N}$, so f is coded in $\mathfrak{N}$ since the n-fold Cartesian product $(c_{n+1})^{n+1}$ exists in N as arranged earlier. By Lemma 4.4, there is some $s \in N$, such that for all $m \in M$, $f(m) \in M$ iff $f(m) \trianglelefteq s$. Now use GW^* to find $b \in N$ such that $\mathfrak{N} \models b = \{x : x \lhd s\}$, and define:

$$\theta_{\exists v \delta(v, \overrightarrow{x})} := \exists v \in b \ \theta_\delta(v, \overrightarrow{x}).$$

□

We are now in possession of sufficient machinery to present:

Proof of Theorem B. By Lemma 4.3 $\mathfrak{M}$ satisfies $EST(\mathcal{L})+$ Power Set $+$ GW^*. Hence we only need to verify:

(1) $\Sigma_\infty(\mathcal{L}^*)$-Replacement, where $\mathcal{L}^* = \{\in\} \cup \{S : S \in \mathcal{A}\}$, and
(2) **Ord** is weakly compact.

To establish $\Sigma_\infty(\mathcal{L}^*)$-Replacement, suppose that for some $\mathcal{L}^*$-formula $\varphi(x, y)$ (with suppressed parameters), and some $a \in M$,

$(*)$ $(\mathfrak{M}, S)_{S \in \mathcal{A}} \models \forall x \in a \ \exists! y \ \varphi(x, y)$.

We first wish to find a $\lhd$-initial segment of N containing M on which the formula $\theta_\varphi(x, y)$ (whose parameters are also suppressed) defines the graph of a function whose domain is a subset of a. This can easily be accomplished by $\Delta_0(\mathcal{L})$-overspill: fix $d \in N$ with $M \subseteq d_F$ and consider the $\Delta_0(\mathcal{L})$-formula $\psi(v)$:

$$\psi(\mathbf{v}) := \forall x \in a \ [\exists^{\leq 1} y \in d \ (y \lhd \mathbf{v} \text{ and } \theta_\varphi(x, y))].$$

By $(*)$ and Lemma 4.5, every element of M is a solution to ψ. Hence by $\Delta_0(\mathcal{L})$-overspill there is some $a \in N \backslash M$ satisfying ψ. Let $c = \{t \in d : t \lhd a\}$ in the sense of $\mathfrak{N}$. Note that:

$$\text{if } m_0 \in a_E,\ m_1 \in c_F, \text{ and } \mathfrak{N} \models \theta_\varphi(m_0, m_1), \text{ then } m_1 \in M.$$

Hence

$$\{y \in c_F : \mathfrak{N} \models \exists x \in a \ \theta_\varphi(x, y)\} = \{y \in M : \mathfrak{M} \models \exists x \in a \ \varphi(x, y)\}.$$

By $\Delta_0(\mathcal{L})$-Separation in $\mathfrak{N}$, the left hand side of the above equation is the extension of some element $m \in N$. But since $m_F \subseteq M$, by part (2) of Lemma 4.2, $m \in M$. This completes the verification of $\Sigma_\infty(\mathcal{L}^*)$-Replacement in $\mathfrak{M}$.

We now know that $(\mathfrak{M}, S)_{S \in \mathcal{A}}$ is a model of GBC. To show that **Ord** is weakly compact in $(\mathfrak{M}, \mathcal{A})$, we need to refine the reasoning used in the proof of (3) $\Rightarrow$ (1) of Lemma 3.3 in order to use $\Delta_0(\mathcal{L})$-overspill instead of elementarity. Suppose τ is an **Ord**-tree coded in $\mathcal{A}$. Thanks to the existence of a global well-ordering in $\mathcal{A}$ we may assume without loss of generality that $\tau = (M, <_\tau)$ for some relation $<_\tau$ of the form $r \cap M$, where $r \in N$, and

$$(\mathfrak{M}, \tau) \models \forall x, y \ (x <_\tau y \rightarrow x \lhd y).$$

Fix $c \in N$ with $M \subseteq c_E$. Let k be the field of r, and consider the relational structure $\tau^* := (k, <_{\tau^*}) \in N$. For $x \subseteq k$, let $\tau^*(x) = (x,\ x^2 \cap <_{\tau^*})$, and let $\theta(\mathbf{v})$ be the $\Delta_0(\mathcal{L})$-formula:

$\exists z \in c$ ["$z = \{y : y \lhd \mathbf{v}\}$" and "$\tau^*(z)$ is a tree" and $\forall x, y \in z \ (x <_{\tau^*} y \rightarrow x \lhd y)$].

Clearly for all $m \in M, \mathfrak{N} \models \theta(m)$. Therefore by $\Delta_0(\mathcal{L})$-overspill there is an element of $N \backslash M$ satisfying φ. This shows that there is some initial segment $\overline{\tau}$ of τ^* in N whose field $\overline{k}$ contains M, and $<_{\overline{\tau}}$ is stronger than $\lhd$ on $\overline{k}$. It follows that $\overline{\tau}$ does not introduce any new elements below the elements of τ (in the sense of the tree ordering). So we can choose $t \in \overline{k} \backslash M$, and define the desired branch $B \in \mathcal{A}$ of τ by $B := \{m \in M : m <_{\overline{\tau}} t\}$. □

Remark 4.5. It is shown in [E-4] that, similar to Theorem A, Theorem 1.1 also has a reversal: If $\mathfrak{N}$ is a model of the fragment $I\text{-}\Delta_0$ of PA (also known as *bounded arithmetic*) that has an automorphism whose fixed point set is a proper initial segment $\mathfrak{M}$ of $\mathfrak{N}$, then $\mathfrak{M}$ is a model of PA.

Remark 4.6. By a theorem of Kaye, Kossak, and Kotlarski [KKK, Theorem 5.6], for a countable recursively saturated model $\mathfrak{M}$ of PA, the condition "I is a strong elementary cut of $\mathfrak{M}$" is equivalent to the condition "there is an automorphism of $\mathfrak{M}$ whose fixed point set is precisely I". The analogue of this result for models of set theory is established in [E-5].

5. CONSEQUENCES FOR NFU

In this section we reap the benefits of Theorems A and B for the Quine-Jensen system of set theory NFU. The theory NFU was introduced by Jensen [Jen] as a modification of Quine's elegant formulation NF (New Foundations) [Q] of Russell's theory of types. NF is a first order theory formulated in the usual language of set theory $\{\in\}$ whose axioms consist of the *stratifiable* comprehension scheme and the usual extensionality axiom. The stratifiable comprehension scheme is the collection of sentences of the form "$\{x : \varphi(x)\}$ exists", provided there is an integer valued function f whose domain is the set of all variables occurring in φ, which satisfies the following two requirements:

1. $f(v) + 1 = f(w)$, whenever $(v \in w)$ is a subformula of φ;
2. $f(v) = f(w)$, whenever $(v = w)$ is a subformula of φ.

Since the formula $\varphi(x) := (x = x)$ is stratifiable, NF proves the existence of a universal set, but the stratification requirement allows NF to avoid - at least seemingly - the usual paradoxes of the existence of "large sets" such as Russell's and Burali-Forti's. However, the formal consistency of NF relative to any ZF-style system of set theory remains an open question.

Jensen's variant NFU of NF is obtained by modifying the extensionality axiom so as to allow *urelements* (hence the U in NFU). Inspired by Specker's work [Sp] on the equiconsistency of NF with the simple theory of types augmented by the ambiguity scheme, Jensen made a breakthrough by establishing the consistency of NFU relative to the fragment $ZBQC$[9] of Zermelo set theory consisting of EST + Power Set. Jensen's method, as refined by Boffa [Bo], shows that one can construct a model of

$$NFU^+ := NFU + \text{Choice} + \text{Infinity},$$

starting from a model $\mathfrak{M}$ of $ZBQC$ that has an automorphism j with $j(\kappa) \geq (2^\kappa)^{\mathfrak{M}}$ for some infinite cardinal κ of $\mathfrak{M}$. In the other direction, Hinnion [Hi] showed that in every model of $\mathfrak{A}$ of NFU^+ one can uniformly interpret a Zermelian structure $Z^{\mathfrak{A}}$ of $ZFC\backslash\{\text{Power Set}\}$, and a nontrivial endomorphism k of $Z^{\mathfrak{A}}$ onto a proper initial

[9] Zermelo set theory with bounded quantification was independently discovered by MacLane who dubbed it $ZBQC$ and championed it as a parsimonious foundation for mathematical practice [Mac, p.373]. Curiously, $ZBQC$ was also isolated by Ressayre [Re], in his work on the model theory of weak systems of set theory (Ressayre calls this theory *Bounded Set Theory, BST*). Mathias' [Mat] is an excellent source of information about $ZBQC$.

segment of $Z^{\mathfrak{A}}$. Hinnion's work on Z was rediscovered by Holmes [Ho-1], whose account prompted Solovay [So] to provide an alternative streamlined development of Z. The endomorphism k can be used to "unravel" $Z^{\mathfrak{A}}$ to a model $(Z^*)^{\mathfrak{A}}$ of $ZBQC$ that has a nontrivial automorphism j. Consequently, the construction of models of the radical system NFU is reduced to building appropriate automorphisms of models of fragments of orthodox set theory. To summarize:

Theorem 5.1. (Jensen-Boffa-Hinnion) *NFU^+ has a model iff there is a model $\mathfrak{M}$ of $ZBQC$ that has an automorphism j such that for some infinite cardinal κ of $\mathfrak{M}$, $(2^\kappa)^{\mathfrak{M}} \leq j(\kappa)$.*

The system $NFUA$ is obtained by augmenting the theory NFU^+ with the axiom[10] "every Cantorian set is strongly Cantorian". A set X is said to be *Cantorian* if there exists a bijection between X and the set of its singletons $UCS(X) := \{\{x\} : x \in X\}$. *Strongly Cantorian* sets, on the other hand, are sets X for which the graph of the "obvious" bijection $x \longmapsto \{x\}$ between X and $UCS(X)$ forms a set. In the framework of the usual Zermelo-style systems of set theory, every nonempty set is obviously strongly Cantorian. However, in NFU the universal set V and many other "large" sets are not Cantorian. Moreover, there are models of NFU in which the set of finite cardinals provides an example of a Cantorian set that fails to be strongly Cantorian. The following result shows that building models of $NFUA$ amounts to building appropriate automorphisms of models of the theory $ZBQC(\{\in, \lhd\}) + GW^*(\lhd)$.

Theorem 5.2. (Folklore). *Let $T_0 := ZBQC(\{\in, \lhd\}) + GW^*(\lhd)$.*
1. *If $\mathfrak{A} \models NFUA$, then there is an initial segment $\mathfrak{N}$ of $(Z^*)^{\mathfrak{A}}$ satisfying T_0 such that the collection of Cantorian elements $(CZ)^{\mathfrak{A}}$ of $Z^{\mathfrak{A}}$ form a $\lhd$-cut of $\mathfrak{N}$, and there is an automorphism j of $\mathfrak{N}$ whose fixed point set is $(CZ)^{\mathfrak{A}}$.*

2. *If $\mathfrak{N} \models T_0$ has an automorphism j fixing a $\lhd$-cut $\mathfrak{M}$ of $\mathfrak{N}$, and for some infinite cardinal κ, $(2^\kappa)^{\mathfrak{M}} \leq j(\kappa)$, then there is a model $\mathfrak{A}$ of $NFUA$ such that $\mathfrak{M}$ is isomorphic to $(CZ)^{\mathfrak{A}}$.*

Holmes introduced an extension $NFUM$ of $NFUA$ in [Ho-1] and proved in [Ho-2] that $NFUM$ is consistent relative to ZFC + "there is a measurable cardinal". This provided an upper bound for the consistency strength of $NFUA$ until Solovay established the unexpected calibrations of (1) the consistency strength of $NFUA$ in terms of Mahlo cardinals [1995, unpublished], and (2) the consistency strength of the system $NFUB$ in terms of weakly compact cardinals [So] ($NFUB$ is an intermediate system between $NFUA$ and $NFUM$). The next result uses Theorems A and B to reveal an intimate connection between $ZFC + \Phi$ and $NFUA$.

Theorem 5.3. *The following are equivalent for a theory T in the language $\{\in\}$:*

1. *T is a consistent completion of $ZFC + \Phi$.*
2. *There is a model $\mathfrak{A}$* of *$NFUA$ such that $T = Th(CZ)^{\mathfrak{A}}$.*

Proof. For (1) $\Rightarrow$ (2), suppose T is a consistent completion of $ZFC + \Phi$. By Theorem A there is a model $\mathfrak{M} = (M, E, \lhd)$ of T such that $\mathfrak{M} \models ZF(\lhd) + GW$, and $\mathfrak{M}$ has a proper elementary $\lhd$-end extension $\mathfrak{N}$ such that M is the fixed point set of some automorphism of $\mathfrak{N}$. So by part (2) of Theorem 5.2, there is a model

[10] This axiom was first considered by Henson [He].

$\mathfrak{A}$ of $NFUA$ such that $T = Th(CZ)^{\mathfrak{A}}$. (2) $\Rightarrow$ (1) is an immediate consequence of coupling part (1) of Theorem 5.2 with Theorem B. $\square$

Corollary 5.4. (Solovay for $\mathbf{L}^{CZ}$, Holmes [Ho-2] for CZ) *$NFUA$ proves the existence of n-Mahlo cardinals in CZ.*

Corollary 5.5. (Solovay) *The following theories are equiconsistent:*

1. $T_1 := ZFC + \{$"*there is an n-Mahlo cardinal*": $n \in \omega\}$.
2. $T_2 := NFUA$.

Proof. To see that $\mathrm{Con}(T_1) \Rightarrow \mathrm{Con}(T_2)$, assume that T_1 has a model $\mathfrak{M}$. By Proposition ♦ of Section 2, $ZFC + \Phi$ is finitely satisfiable in $\mathfrak{M}$ and therefore $ZFC + \Phi$ is consistent. Now use (1) $\Rightarrow$ (2) of Theorem 5.3. The implication $\mathrm{Con}(T_2) \Rightarrow \mathrm{Con}(T_1)$ is an immediate consequence of (2) $\Rightarrow$ (1) of Theorem 5.3. $\square$

Our last corollary, which follows directly from Corollaries 2.1.1 and 2.1.2 and Theorem 5.3, shows that if $NFUA$ is consistent, then it cannot settle the truth value of many statements, such as $\mathbf{V} = \mathbf{L}$ or Continuum Hypothesis, in CZ.

Corollary 5.6. *Assume $NFUA$ is consistent. If ψ is a statement of set theory satisfying either*
1. $ZF \vdash$ "*ψ holds in $\mathbf{L}$*", *or*
2. $ZFC \vdash$ "*for some poset $\mathbb{P}$, $1_{\mathbb{P}} \Vdash \psi$*",

then there is a model $\mathfrak{A}$ of $NFUA$ such that $(CZ)^{\mathfrak{A}} \models \psi$.

Remark 5.7. Part 1 of Theorem 5.2, Theorem B, and [E-2, Theorem 3.6] together imply that there are countable models of $ZFC + \Phi$ that are not isomorphic to any model of the form $(CZ)^{\mathfrak{A}}$, with $\mathfrak{A} \models NFUA$. However, every *countable recursively saturated* model of $ZFC + \Phi$ is of the form $(CZ)^{\mathfrak{A}}$, for some $\mathfrak{A} \models NFUA$. This can be verified by coupling Theorem 5.3 with the classical theorem of Barwise and Ressayre [D, Theorem 4.59] on the resplendence property of countable recursively saturated models. Conversely, it can be shown (using Theorem B, Theorem 5.2, and [E-3, Theorem 4.5(i)]) that if $\mathfrak{A} \models NFUA$ and $\omega^{\mathfrak{A}}$ is nonstandard, then $(CZ)^{\mathfrak{A}}$ is recursively saturated.

References

[Bar] J. Barwise, **Admissible Sets and Structures**, Springer-Verlag, Berlin (1975).
[Bau] J. Baumgartner, *Canonical partition relations*, **J. Sym. Logic**, vol. 40 (1975), pp. 541-554.
[Bo] M. Boffa, *ZFJ and the consistency problem for NF*, **Jahrbuch der Kurt-Gödel-Gesellschaft**, pp. 102-106 (1988).
[D] K. Doets, **Basic Model Theory**, CSLI publications, Stanford (1996).
[E-1] A. Enayat, *On certain elementary extensions of models of set theory*, **Trans. Amer. Math**. Soc. vol. 283 (1984), pp. 705-715.
[E-2] _____, *Conservative extensions of models of set Theory and generalizations*, **J. Sym. Logic**, vol. 51 (1986), pp. 1005-1021.
[E-3] _____, *Power-like models of set theory*, **J. Sym. Logic**, vol. 66 (2001), pp.1766-1782.
[E-4] _____, *From bounded to second order arithmetic via automorphisms*, in preparation.
[E-5] _____, *Automorphisms of models of set theory*, in preparation.
[ER] P. Erdös and R. Rado, *A combinatorial theorem*, **J. London Math. Society**, vol. 25 (1950), pp. 249-255.
[Fe] U. Felgner, *Comparisons of the axioms of local and global choice*, **Fund. Math**. vol. 71 (1971), pp. 43-62.
[Fo] Forster, **Set Theory with a Universal Set**, Second ed., Oxford Logic Guides, vol. 31, Oxford University Press (1995).

[G] H. Gaifman, *Models and types of arithmetic,* **Ann. Math. Logic**, vol. 9 (1976), pp. 223-306.

[He] W. Henson,*Type-raising operations on cardinals and ordinals in Quine's "New Foundations"*, **J. Sym. Logic**, vol.39 (1973), pp. 59-68 .

[Hi] R. Hinnion, **Sur la théorie des ensembles de Quine**, Ph. D. thesis, UMD Brussels (1975).

[Ho-1] R. Holmes, **Elementary Set Theory with a Universal Set**, Cahiers du Centre de Logic, vol. 10, Academia, Louvain la-Neuve, Belgium (1998).

[Ho-2] _____, *Strong axioms of infinity in NFU,* **J. Sym. Logic**, vol. 66 (2001), pp. 87-116.

[Jec] T. Jech, **Set Theory**, Academic Press, New York (1978).

[Jen] B. Jensen, *On the consistency of a slight (?) modification of Quine's New Foundations,* **Synthese**, vol. 19 (1969), pp. 250-263 .

[Kan] A. Kanamori, **The Higher Infinite**, Springer-Verlag, Berlin (1994).

[Kau] M. Kaufmann, *Blunt and topless end extensions of models of set theory*, **J. Sym. Logic** 48 (1983), pp. 1053-1073.

[KKK] R. Kaye, R. Kossak, and H. Kotlarski, *Automorphisms of recursively saturated models of arithmetic*, **Ann. Pure Appl. Logic**, vol. 55 (1991), pp.67-99.

[KP] L. Kirby and J. Paris, *Initial segments of models of Peano's axioms*, in **Lecture Notes in Mathematics**, vol. 619, Springer-Verlag, 1977, pp. 211-226.

[Ku] K. Kunen, *Some applications of iterated ultrapowers in set theory*, **Ann. Math. Logic** 1 (1970), pp. 179-227.

[Mac] S. MacLane, **Mathematics, Form and Function**, Springer-Verlag (1986).

[Mat] A. Mathias, *The strength of Mac Lane set theory*, vol. 110 (2001), no. 1-3, pp. 107–234.

[Q] W.V.O. Quine, *New foundations for mathematical logic*, **Amer. Math. Monthly**, vol. 44 (1937), pp. 70-80.

[Re] J. P. Ressayre, *Modèles non standard et sous-systèm remarquable de ZF*, in **Modeles Non Standard en Arithmetic et Theorie des Ensembles** (J. P. Ressayre and A. Wilke, ed.), Publication Mathematiques de l'Universite Paris 7, Paris (1986).

[Ro] J.B. Rosser, **Logic for Mathematicians**, McGraw-Hill, reprinted (with appendices) by Chelsea, New York (1978).

[SS] J. Schmerl and S. Shelah, *On power-like models for hyperinaccessible cardinals*, **J. Sym. Logic** 37 (1972), pp. 531-537

[So] R. Solovay, *The consistency strength of NFUB,* preprint available at Front for the Mathematics ArXiv, http://front.math.ucdavis.edu

[Sp] E. Specker, *Typical ambiguity*, **Logic, Methodology, and Philosophy of Science** (E. Nagel, ed.), Stanford (1962).

Department of Mathematics and Statistics, American University, Washington, DC, 20016-8050

E-mail address: enayat@american.edu