

AN INEVITABLE EXTENSION OF ZFC

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Mahlo Cardinals

- A *Mahlo* cardinal κ is a strongly inaccessible cardinal such that the regular cardinals below κ form a *stationary* subset of κ .
- For an ordinal α , the α -*Mahlo cardinals* are defined recursively as follows:

κ is 0-Mahlo if κ is strongly inaccessible;

For $\alpha = \delta + 1$, κ is a α -Mahlo if

$\{\gamma < \kappa : \gamma \text{ is } \delta\text{-Mahlo}\}$ is stationary in κ ;

For limit α , κ is α -Mahlo if κ is δ -Mahlo for all $\delta < \alpha$.

Levy and Reflection

- Levy showed that Σ_n -truth is Σ_n -definable for $n \geq 1$ within ZF.
- In particular, for each natural number n there is a unary formula with the free variable α , denoted “ $V_\alpha \prec_n \mathbf{V}$ ”, that expresses:

for all Σ_n -formulae $\varphi(v_1, \dots, v_k)$, and all a_1, \dots, a_k in V_α ,

$$\varphi(a_1, \dots, a_k) \leftrightarrow \varphi^{V_\alpha}(a_1, \dots, a_k).$$

- For a unary formula $C(\alpha)$, possibly with suppressed parameters,

“ $\{\alpha : C(\alpha)\}$ is c.u.b.”

stands for the formula expressing

“ $\{\alpha \in \mathbf{Ord} : C(\alpha)\}$ is c.u.b in \mathbf{Ord} ”.

Levy and Reflection, cont'd

- **Reflection Theorem** (Montague 1957; Levy 1960) *For each natural number n , ZF proves that $\{\alpha : V_\alpha \prec_n \mathbf{V}\}$ is c.u.b.*
- **Theorem** (Levy 1960). *For each natural numbers n , the following statement is provable within ZF:*

 $(\kappa \text{ is } (n+1)\text{-Mahlo}) \rightarrow \exists \alpha < \kappa (\alpha \text{ is } n\text{-Mahlo and } V_\alpha \prec V_\kappa).$

The Levy Scheme Λ

- $\lambda_{m,n}(\kappa)$ is the sentence in set theory asserting that κ is an m -Mahlo cardinal and

$$V_\kappa \prec_n \mathbf{V}.$$

- $\Lambda := \{\exists \kappa (\lambda_{n,n}(\kappa) : n \in \omega)\}.$
- $\Lambda_1 := \{\forall \alpha \in \mathbf{Ord} \exists \kappa > \alpha \lambda_{n,n}(\kappa) : n \in \omega\}.$
- $\Lambda_2 := \{\forall \alpha \in \mathbf{Ord} \exists \kappa > \alpha \lambda_{m,n}(\kappa) : m \in \omega, n \in \omega\}.$
- $\Lambda_3 := \{\psi_{C(\alpha,x),n} : C = C(\alpha, x) \text{ is a binary formula of set theory}\},$ where

$$\begin{aligned} \psi_{C,n} &:= \forall x [\{\alpha \in \mathbf{Ord} : C(\alpha, x)\} \text{ is c.u.b.} \\ &\quad \rightarrow \exists \kappa C(\kappa, x) \text{ and } \kappa \text{ is } n\text{-Mahlo}]. \end{aligned}$$

Different Faces of Λ

- **Theorem** (Levy 1960). *Over ZF, the theories Λ , Λ_1 , Λ_2 , and Λ_3 are pairwise equivalent.*
- $\Lambda_0 := \{\exists \kappa \text{ } \kappa \text{ is } n\text{-Mahlo: } n \in \omega\}$.
- **Proposition** (Folklore)
 - (a) *The theories $ZF + \Lambda_0$ and $ZF + \Lambda$ are equiconsistent.*
 - (b) *Moreover, assuming $\text{Con}(ZF + \Lambda_0)$, neither Λ_0 , nor Λ is finitely axiomatizable over ZF.*

The robustness of Λ

- **Theorem** *If $M \models \text{ZFC} + \Lambda$, and $c \in M$, then $(\mathbf{L}(c))^M \models \Lambda$.*
- **Theorem** *If $M \models \text{ZFC} + \Lambda$ and $\mathbb{P} \in M$ is a partial order, then for every \mathbb{P} -generic filter G over M , $M[G] \models \Lambda$.*
- **Corollary.** *Suppose $\text{Con}(\text{ZF} + \Lambda)$. Then for any sentence ψ , $\text{Con}(\text{ZF} + \Lambda + \psi)$ if at least one of the following conditions are true:

(a) $\text{ZF} \vdash$ “ ψ holds in \mathbf{L} ”, or

(b) $\text{ZF} \vdash$ “for some poset \mathbb{P} , $1_{\mathbb{P}} \Vdash \psi$ ”,*

Finite Set Theory

- $TC :=$ “every set has a transitive closure”.
- $ZF_{\text{fin}} = ZF \setminus \{\text{Infinity}\} + \neg\text{Infinity} + TC.$
- $GBC_{\text{fin}} = GBC \setminus \{\text{Infinity}\} + \neg\text{Infinity} + TC.$
- **Theorem** [Ackermann 1940, Kaye-Wong 2008]
 - (a) ZF_{fin} is bi-interpretable with PA.
 - (b) GBC_{fin} is bi-interpretable with ACA_0 .

Inevitability of Λ , Exhibit 1

- Let “Ord is WC” be the statement in class theory asserting that every “**Ord**-tree” has a branch of length **Ord**.
- **Theorem** [E 2004]
 - (a) *If $(M, \mathcal{A}) \models \text{GBC} + \text{Ord is WC}$, then $M \models \text{ZFC} + \Lambda$.*
 - (b) *Every completion of $\text{ZFC} + \Lambda$ has a countable model that has an expansion to a model of $\text{GBC} + \text{Ord is WC}$.*
- **Corollary** *$\text{GBC} + \text{Ord is WC}$ is a conservative extension of $\text{ZFC} + \Lambda$.*
- **Theorem** (Folklore) *GBC_{fin} is a conservative extension of ZF_{fin} .*

Inevitability of Λ , Exhibit 2

- ZFC(I) is a theory in the language $\{\in, \mathbf{I}(x)\}$, where $\mathbf{I}(x)$ is a unary predicate.
- The axioms of ZFC(I) are as follows.
 - (1) ZFC + All instances of replacement (hence separation) in $\{\in, \mathbf{I}(x)\}$;
 - (2) \mathbf{I} is a cofinal subclass of ordinals;
 - (3) \mathbf{I} is a class of indiscernibles for (\mathbf{V}, \in) .

Exhibit 2, Continued

- **Theorem** (E 2005). *The following are equivalent for a completion T of ZFC:*
 - (1) *T has a model M that expands to a model $(M, \mathbf{I}) \models \text{ZFC}(\mathbf{I})$.*
 - (2) *T has a model M that expands to $(M, \mathbf{I}_n)_{n < \omega}$ satisfying $\text{ZF}(\{\mathbf{I}_n : n \in \omega\}) + “\mathbf{I}_{n+1}$ is a set of indiscernibles for $(\mathbf{V}, \mathbf{I}_k)_{k \leq n}”$.*
 - (3) *T is an extension of $\text{ZFC} + \Lambda$.*
- **Remark.** If Replacement (I) is weakened to Separation(I), the resulting system is conservative over ZFC.
- **Theorem** [E 2005] $\text{ZF}_{\text{fin}}(\mathbf{I})$ is a conservative extension of ZF_{fin} .

Inevitability of Λ , Exhibit 3

- **Theorem.** [E 2001] *The Continuum Hypothesis is a sufficient, but not a necessary condition for every consistent extension of ZF to have an \aleph_2 -like model.*
- **Theorem** [Kaufmann, E 1984] *Every completion of ZFC has a θ -like model for every $\theta \geq \aleph_1$.*
- **Theorem.** [E 2001] *Con(ZF + there is an ω -Mahlo cardinal) implies consistency of “the only completions of ZFC that have an \aleph_2 -like model are those containing Λ ”.*
- **Theorem** (McDowell-Specker 1961). *Every completion of ZF_{fin} has a θ -like model for every $\theta \geq \aleph_1$.*

Inevitability of Λ , Exhibit 4

- The theory NFU was introduced by Jensen as a modification of Quine's elegant formulation NF (New Foundations) of Russell's theory of types.
- NF is a first order theory whose axioms consist of the *stratifiable* comprehension scheme and the usual extensionality axiom.
- The stratifiable comprehension scheme is the collection of sentences of the form “ $\{x : \varphi(x)\}$ exists”, provided there is an integer valued function f whose domain is the set of all variables occurring in φ , which satisfies the following two requirements: (1) $f(v) + 1 = f(w)$, whenever $(v \in w)$ is a subformula of φ ; (2) $f(v) = f(w)$, whenever $(v = w)$ is a subformula of φ .

Exhibit 4, Continued

- Jensen's variant NFU of NF is obtained by modifying the extensionality axiom so as to allow *urelements*.
- **Theorem** (Jensen 1968)
 - (a) $\text{Con}(\text{PA}) \Rightarrow \text{Con}(\text{NFU} + \neg\text{Infinity})$.
 - (b) $\text{Con}(\text{Z}) \Rightarrow \text{Con}(\text{NFU} + \text{Choice} + \text{Infinity})$
- X is *Cantorian* if there is a one-to-one correspondence between X and $\{\{v\} : v \in X\}$; X is *strongly Cantorian* if the map sending v to $\{v\}$ (as v varies in X) exists;
- $\text{H} :=$ "every Cantorian set is strongly Cantorian"
- $\text{NFUA} := \text{NFU} + \text{Infinity} + \text{Choice} + \text{H}$.
- $\text{NFUA}_{\text{fin}} := \text{NFUA} \setminus \{\text{Infinity}\} + \{\neg\text{Infinity}\}$.

Exhibit 4, Continued

- **Theorem** (Solovay, 1995) $\text{Con}(\text{ZFC} + \Lambda_0) \Leftrightarrow \text{Con}(\text{NFUA})$.
- **Theorem** (E 2002) *The following are equivalent for a theory T in the language $\{\in\}$:*
 - (a) *T is a consistent completion of $\text{ZFC} + \Lambda$.*
 - (b) *There is a model M of NFUA such that $T = \text{Th}(\text{"Cantorian part of } V\text{"})^M$.*
- **Theorem** (Solovay-E, 2002). *The analogue of the above theorem holds for ZF_{fin} and NFUA_{fin} , in particular:*

$$\text{Con}(\text{NFUA}_{\text{fin}}) \Leftrightarrow \text{Con}(\text{ZF}_{\text{fin}}).$$

Inevitability of Λ , Exhibit 5

- EST is ZFC \setminus {Power Set, Replacement} + Δ_0 -Separation.
- GW is the axiom in the language $\{\in, \triangleleft\}$ that is the conjunction of the following 4 axioms:
 - (1) \triangleleft totally orders the universe;
 - (2) Every nonempty set has a \triangleleft -least element,
 - (3) $x \in y \rightarrow x \triangleleft y$;
 - (4) $\forall x \exists y \forall z (z \in y \iff z \triangleleft x)$.

Exhibit 5, Continued

- **Theorem** [E 2004]

(a) *For every completion T of $ZFC + \Lambda$ there is a model M_0 of $T + ZF(\triangleleft) + GW$ such that M_0 has a proper e.e.e. M such that for some automorphism f of M , the fixed point set of f is M_0 .*

(b) *Moreover, if j is an automorphism of $M \models EST$ whose fixed point set M_0 is a \triangleleft -initial segment of N , then $M_0 \models ZFC + \Lambda$.*

- **Theorem**

(a) (Gaifman) *The analogue of (a) above for ZF_{fin} .*

(b) (E 2004) *The analogue of (b) above for ZF_{fin} (with $I-\Delta_0$ instead of EST).*