

**WHAT NFU KNOWS ABOUT
CANTORIAN OBJECTS**

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BIRTH OF NFU

- Jensen's variant NFU of NF is obtained by modifying the extensionality axiom so as to allow *urelements*.
- $NFU^+ := NFU + \text{Infinity} + \text{Choice}$.
- $NFU^- := NFU + \text{"V is finite"} + \text{Choice}$.
- **Theorem** (Jensen, 1968) *If (a fragment of) Zermelo set theory is consistent, then so are NFU^+ and NFU^- . Moreover, if ZF has an α -standard model, then so does NFU^+ .*

NFU and Orthodox Set Theory (1)

- $EST(\mathcal{L})$ [Elementary Set Theory] is obtained from $ZFC(\mathcal{L})$ by deleting Power Set and Replacement, and adding $\Delta_0(\mathcal{L})$ -Separation. More explicitly, it consists of Extensionality, Foundation (every nonempty set has an \in -minimal member), Pairs, Union, Infinity, Choice, and $\Delta_0(\mathcal{L})$ -Separation.
- GW_0 [Global Well-ordering] is the axiom in the language $\mathcal{L} = \{\in, \triangleleft\}$, expressing “ \triangleleft well-orders the universe”.
- GW is the strengthening of GW_0 obtained by adding the following two axioms to GW_0 :
 - (a) $\forall x \forall y (x \in y \rightarrow x \triangleleft y)$;
 - (b) $\forall x \exists y \forall z (z \in y \leftrightarrow z \triangleleft x)$.
- $ZBQC = EST + \text{Power Set}$.

NFU⁺ and Orthodox Set Theory (2)

- Jensen's method, as refined by Boffa, shows that one can construct a model of NFU^+ starting from a model \mathfrak{M} of $ZBQC$ that has an automorphism j with $j(\kappa) \geq (2^\kappa)^\mathfrak{M}$ for some infinite cardinal κ of \mathfrak{M} .
- In the other direction, Hinnion and later, Holmes showed that in NFU^+ one can interpret (1) a 'Zermelian structure' Z that satisfies $ZFC \setminus \{\text{Power Set}\}$, and (2) a nontrivial endomorphism k of Z onto a proper initial segment of Z .
- The endomorphism k can be used to "unravel" Z to a model \bar{Z} of $ZBQC$ that has a nontrivial automorphism j .

Large Cardinals and NFU (1)

- X is *Cantorian* if there exists a bijection between X and the set of its singletons $UCS(X) := \{\{x\} : x \in X\}$.
- X is *strongly Cantorian* if the graph of the “obvious” bijection $x \mapsto \{x\}$ between X and $UCS(X)$ forms a set.
- Cantorian elements of models of *NFU* correspond to fixed points of automorphisms of models of *ZF*.

Large Cardinals and NFU (2)

- Holmes introduced an extension $NFUM^+$ of NFU^+ which imposes powerful closure conditions on the strongly Cantorian parts of Z , and showed the equiconsistency of $NFUM^+$ and KMC plus “the class of ordinals is a measurable cardinal”.
- $NFUM^+$ has two distinguished fragments: $NFUA^+$ and $NFUB^+$.
- $NFUA^\pm := NFU^\pm$ plus “every Cantorian set is strongly Cantorian”.

Large Cardinals and NFU (3)

- Solovay showed:

(1) $NFUA^+$ is equiconsistent with $ZFC + \{\text{“there is an } n\text{-Mahlo cardinal”} : n \in \omega\}$.

(2) $NFUB^+$ is equiconsistent with KM plus “the class of ordinals is weakly compact”.

- **Question:** What does $NFUA^+$ know about the Cantorian part CZ of Z ?

- **Answer** (Solovay-Holmes): At least ZFC plus

$\{\text{“there is an } n\text{-Mahlo cardinal”} : n \in \omega\}$.

Large Cardinals and Automorphisms (1)

- Let Φ be

$$\{\exists \kappa (\kappa \text{ is } n\text{-Mahlo and } V_\kappa \prec_{\Sigma_n} \mathbf{V}) : n \in \omega\}.$$

- Over ZFC , Φ is stronger than, but equiconsistent with

$$\{\text{“there is an } n\text{-Mahlo cardinal”} : n \in \omega\}.$$

- **Theorem.** *Suppose T is a consistent completion of $ZFC + \Phi$. There is a model \mathfrak{M} of $T + ZF(\triangleleft) + GW$ such that \mathfrak{M} has a proper e.e.e. \mathfrak{M}^* that possesses an automorphism whose fixed point set is M .*

Large Cardinals and Automorphisms (2)

- **Theorem.** *GBC + “Ord is weakly compact” is a conservative extension of ZFC + Φ .*
- Suppose M is an \triangleleft -initial segment of $\mathfrak{M}^* := (M^*, E, <)$. We define:

$$SSy(\mathfrak{M}^*, M) = \{a_E \cap M : a \in M^*\},$$

where $a_E = \{x \in M^* : x E a\}$.

- **Theorem.** *If j is an automorphism of a model $\mathfrak{M}^* = (M^*, E, <)$ of $EST(\{\in, \triangleleft\}) + GW$ whose fixed point set M is a \triangleleft -initial segment of \mathfrak{M}^* , and $\mathcal{A} := SSy(\mathfrak{M}^*, M)$, then $(\mathfrak{M}, \mathcal{A}) \models GBC + \text{“Ord is weakly compact”}$.*
- **Corollary.** *What $NFUA^+$ knows about CZ is precisely ZFC + Φ .*

Strong Cuts (1)

- M is a *strong* \triangleleft -cut of \mathfrak{M}^* , if M is a \triangleleft -cut of \mathfrak{M}^* and for each function $f \in M^*$ whose domain includes M , there is some s in M , such that for all $m \in M$,

$$f(m) \notin M \quad \text{iff} \quad s \triangleleft f(m).$$

- Suppose \mathfrak{M} is a strong \triangleleft -cut of $\mathfrak{M}^* = (M^*, E, <)$ of $EST(\{\in, \triangleleft\}) + GW$ and $\mathcal{A} := SSy(\mathfrak{M}^*, M)$.
- Let $\mathcal{L}^* = \{\in\} \cup \{S : S \in \mathcal{A}\}$.
- For every \mathcal{L}^* -formula $\varphi(\vec{x}, \vec{S})$, with free variables \vec{x} and parameters \vec{S} from \mathcal{A} , there is some $\Delta_0(\mathcal{L})$ -formula $\theta_\varphi(\vec{x}, \vec{b})$, where \vec{b} is a sequence of parameters from N , such that for all sequences \vec{a} of elements of M

$$(\mathfrak{M}, S)_{S \in \mathcal{A}} \models \varphi(\vec{a}, \vec{S}) \quad \text{iff} \quad \mathfrak{M}^* \models \theta_\varphi(\vec{a}, \vec{b}).$$

On $NFUB^+$

- $NFUB^+$ is an extension of $NFUA^+$, obtained by adding a scheme that ensures that the intersection of every definable class with CZ is coded by some set.
- Holmes has shown that $NFUB^+$ canonically interprets KMC plus “**Ord** is weakly compact”.
- Solovay constructed a model of $NFUB^+$ from a model of KMC plus “**Ord** is weakly compact” in which “ $V = L$ ”.
- We can show that what $NFUB^+$ knows about its canonical Kelley-Morse model is precisely:

KMC plus “**Ord** is weakly compact” plus
Dependent Choice Scheme

WHAT ABOUT NFU^- AND ITS EXTENSIONS?

- Solovay has shown

(1) $(I\Delta_0 + Superexp) \vdash$

$$\text{Con}(NFU^-) \iff \text{Con}(I\Delta_0 + Exp).$$

(2) $(I\Delta_0 + Exp) + \text{Con}(I\Delta_0 + Exp) \not\vdash$

$$\text{Con}(NFU^-).$$

On $NFUA^-$

- **Theorem** *The following two conditions are equivalent for any model \mathfrak{M} of the language of arithmetic:*
 - (a) \mathfrak{M} satisfies PA
 - (b) $\mathfrak{M} = \text{fix}(j)$ for some nontrivial automorphism j of an end extension \mathfrak{N} of \mathfrak{M} that satisfies $I\Delta_0$.
- **Key Lemma.** *If $\mathfrak{M} \models I\Delta_0$ and $j \in \text{Aut}(\mathfrak{M})$ with $\text{fix}(j) \subsetneq_e M$, then $\text{fix}(j)$ is a strong cut of \mathfrak{M} .*
- **Theorem.** *What $NFUA^-$ knows about Cantorian arithmetic is precisely PA .*

NFUA⁻, ACA₀, and VA

- For a cut I of \mathfrak{M} , $SSy(\mathfrak{M}, I)$ is the collection of subsets of I of the form $I \cap X$, where X is a coded subset of M .
- (Visser Arithmetic) $VA := I\Delta_0 +$ “ j is a nontrivial automorphism whose fixed point set is downward closed”.
- **Theorem.** *ACA₀ is faithfully interpretable in VA, and VA is faithfully interpretable in NFUA⁻.*

ON $NFUB^-$

- Z_2 is second order arithmetic (also known as *analysis*), and DC is the scheme of *Dependent Choice*.
- **Theorem.** $NFUB^-$ canonically interprets a model of $Z_2 + DC$. Moreover, every countable model of $Z_2 + DC$ is isomorphic to the canonical model of analysis of some model of $NFUB^-$.
- **Corollary.** What $NFUB^-$ knows about analysis is precisely $Z_2 + DC$.

AMENABLE AUTOMORPHISMS

- For an initial segment M of \mathfrak{M}^* , $SSy(\mathfrak{M}^*, M)$ is the collection of subsets of M of the form $M \cap X$, where X is a subset of M^* that is codes in \mathfrak{M}^* .
- $j \in Aut(\mathfrak{M}^*)$ is M -amenable if the fixed point set of j is precisely M , and for every formula $\varphi(x, j)$ in the language $\mathcal{L}_A \cup \{j\}$, possibly with suppressed parameters from M^* ,

$$\{a \in M : (\mathfrak{M}^*, j) \models \varphi(a, j)\} \in SSy(\mathfrak{M}^*, M).$$

Theorem. *Suppose $(\mathfrak{M}, \mathcal{A})$ is a countable model of $Z_2 + DC$. There exists an e.e.e. \mathfrak{M}^* of \mathfrak{M} that has an M -amenable automorphism j such that $SSy(\mathfrak{M}^*, M) = \mathcal{A}$.*

ON LONGEST INITIAL SEGMENTS OF FIXED POINTS

- For a model \mathfrak{M} of arithmetic and $j \in \text{Aut}(\mathfrak{M})$,

$$I_{fix}(j) := \{m \in M : \forall x \leq m (j(x) = x)\}.$$

- **Theorem.** *For countable \mathfrak{M} , \mathfrak{M} satisfies $I\Delta_0 + B\Sigma_1 + Exp$ iff $\mathfrak{M} = I_{fix}(j)$ for some nontrivial automorphism j of an end extension \mathfrak{N} of \mathfrak{M} that satisfies $I\Delta_0$.*

- $B\Sigma_1$ is the Σ_1 -collection scheme consisting of the universal closure of formulae of the form

$$[\forall x < a \exists y \varphi(x, y)] \rightarrow [\exists z \forall x < a \exists y < z \varphi(x, y)],$$

where φ is a Δ_0 -formula.

CONSEQUENCES FOR NFU^-

- NFU^- knows that strongly cantorion numbers satisfy $I\Delta_0 + B\Sigma_1 + Exp$.
- But Solovay has shown that NFU^- knows more about strongly cantorion numbers.
- **Question.** What is the precise knowledge of NFU^- about strongly cantorion numbers?
- **Question.** What is the precise knowledge of NFU^+ about the strongly cantorion part of Z ?