

Set Theory and Indiscernibles

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LEIBNIZ'S PRINCIPLE OF IDENTITY OF INDISCERNIBLES

- The principle of *identity of indiscernibles*, formulated by Leibniz (1686), states that no two distinct substances exactly resemble each other.
- Leibniz's principle can be construed as prescribing a logical relationship between *objects* and *properties*: any two distinct objects must differ in at least one property. This suggests a model theoretic interpretation:
- Fix a model $\mathfrak{M} = (M, \dots)$ in a language \mathcal{L} , let the "objects" refer to the elements of M , and the "properties" refer to properties that are \mathcal{L} -expressible in \mathfrak{M} via first order formulas with one free variable.

LEIBNIZIAN MODELS

- Let us call a model \mathfrak{M} to be *Leibnizian* iff \mathfrak{M} contains no pair of distinct elements a and b , such that for every first order formula $\varphi(x)$ of \mathcal{L} with precisely one free variable x ,

$$\mathfrak{M} \models \varphi(a) \leftrightarrow \varphi(b).$$

- Any pointwise definable model is Leibnizian, e.g., $(\omega, <)$, (V_ω, \in) , and $(L(\omega_1^{CK}), \in)$.
- Any model $\mathfrak{M} = (M, \dots)$ in a language \mathcal{L} such that $|M| > 2^{|\mathcal{L}| \cdot \aleph_0}$ is *not* Leibnizian.
- Every Leibnizian model is rigid, but *not* vice versa: $(\omega_1, <)$ is rigid but not Leibnizian.

LEIBNIZIAN MODELS, CONT'D

- The field \mathbb{R} of real numbers, and the ring of integers \mathbb{Z} are both Leibnizian, but the field \mathbb{C} of complex numbers is *not*.
- Every Archimedean ordered field is Leibnizian.
- Moreover, Tarski's elimination of quantifiers theorem for real closed fields implies that the *Leibnizian real closed fields are precisely the Archimedean real closed fields*.
- Non-Archimedean Leibnizian ordered fields exist in every infinite cardinality $\leq 2^{\aleph_0}$.

THE LEIBNIZ-MYCIELSKI AXIOM (LM)

- Leibniz's principle cannot be expressed in first order logic, even for countable structures. This is an immediate corollary of Ehrenfeucht-Mostowski's theorem on indiscernibles.
- However, Mycielski (1995) has introduced the following first order axiom (LM) in the language of set theory $\{\in\}$ which captures the spirit of Leibniz's principle for models of set theory:

$$\forall x \forall y [x \neq y \rightarrow \exists \alpha > \max\{\rho(x), \rho(y)\}]$$

$$Th(V_\alpha, \in, x) \neq Th(V_\alpha, \in, y)].$$

- **Theorem** (Mycielski). *A complete extension T of ZF proves LM iff T has a Leibnizian model.*

LM AS A CHOICE PRINCIPLE

- *Kinna-Wagner Selection Principles* (1955)

KW_1 : For every family \mathcal{A} of sets there is a function f such that

$$\forall x \in \mathcal{A} (|x| \geq 2 \rightarrow \emptyset \neq f(x) \subsetneq x).$$

KW_2 : Every set can be injected into the power set of some ordinal.

$$ZF \vdash KW_1 \longleftrightarrow KW_2.$$

- GKW_1 : There is a definable (without parameters) map F such that $F(x) \subsetneq x$ for every x with two or more elements.
- GKW_2 : There is a definable (without parameters) map G such that G injects V into the class of subsets of \mathbf{Ord} .

THE EQUIVALENCE OF LM WITH GLOBAL KM

Theorem. *Suppose M is a model of ZF .
The following are equivalent:*

- (i) *M satisfies GKW_1 .*
- (ii) *M satisfies GKW_2 .*
- (iii) *M satisfies LM .*

COROLLARIES

Corollary. $ZF + LM \vdash KW$.

Corollary. $ZF + V = OD \vdash LM$.

Corollary *In the presence of $ZF + LM$ there is a parameter free definable global linear ordering of the universe.*

Corollary. $ZF + LM$ proves $GC_{<\omega}$ (global choice for collections of finite sets).

Corollary. $ZF + LM$ proves the existence of a definable set of real numbers that is not Lebesgue measurable.

OPEN QUESTIONS

- **Question 1.** (Abramson and Harrington, 1977). Does every completion T of ZF have an uncountable model without a pair of indiscernible ordinals?
- **Question 2** (Schmerl). Is there a model of set theory with a pair of indiscernibles, but not with a *triple* of indiscernibles?

ANTI-LEIBNIZIAN SYSTEMS

$ZFC(I)$ is a theory in the language $\{\in, \mathbf{I}(x)\}$, where $\mathbf{I}(x)$ is a unary predicate [but we shall write $x \in \mathbf{I}$ instead of $\mathbf{I}(x)$], whose axioms are:

- $ZFC +$ All instances of replacement in the language $\{\in, \mathbf{I}(x)\}$;

- \mathbf{I} is a cofinal subclass of ordinals:

$$(\mathbf{I} \subseteq \mathbf{Ord}) \wedge$$

$$\forall x \in \mathbf{Ord} \exists y \in \mathbf{Ord} (x \in y \in \mathbf{I});$$

- For each n -ary formula $\varphi(v_1, \dots, v_n)$ in the language $\{\in\}$,

$$\forall x_1 < \dots < x_n, \forall y_1 < \dots < y_n \text{ from } \mathbf{I}$$

$$\varphi(x_1, \dots, x_n) \leftrightarrow \varphi(y_1, \dots, y_n).$$

ZFC(I) AND LARGE CARDINALS

- If κ is a Ramsey cardinal, then (V_κ, \in) expands to a model of *ZFC(I)*.
- If (L_κ, \in) expands to a model of *ZFC(I)*, and $\text{cf}(\kappa) > \omega$, then $0^\#$ exists.
- If $0^\#$ exists, then \mathbf{L} cannot be expanded to *ZFC(I)*.
- Every well-founded model of *ZFC(I)* satisfies $0^\#$ exists.

THE SYSTEM $ZFC(I^{<\omega})$

- $ZFC(I^{<\omega})$ is a theory in the language

$$\{\in\} \cup \{\mathbf{I}_n(x) : n \in \omega\},$$

where each \mathbf{I}_n is a unary predicate, whose axioms are:

- $ZFC +$ All instances of replacement in the language $\{\in\} \cup \{\mathbf{I}_n(x) : n \in \omega\}$;
- \mathbf{I}_n is a cofinal subclass of ordinals;
- \mathbf{I}_0 is a class of indiscernibles for (\mathbf{V}, \in) , and for $n \geq 0$, \mathbf{I}_{n+1} is a class of indiscernibles for the structure $(\mathbf{V}, \in, \mathbf{I}_0, \dots, \mathbf{I}_n)$.
- **Question.** *What are the consequences of $ZFC(I)$ and $ZFC(I^{<\omega})$ in the \in -language of set theory?*

THE ANSWER

- **Theorem.** *The following are equivalent for a completion T of ZFC:*
 1. *Some model of T expands to a model of $ZFC(I)$.*
 2. *Some model of T expands to a model of $ZFC(I^{<\omega})$.*
 3. *Some model of T expands to a model of $GBC + \text{“Ord is weakly compact”}$.*
 4. *T is a completion of $ZFC + \Phi$.*
- $GBC =$ Gödel-Bernays class theory.
- **“Ord is weakly compact”** is the statement **“every Ord-tree has a branch”**.

THE CANONICAL SET THEORY

$ZFC + \Phi$

- $\Phi := \{ \exists \theta (\theta \text{ is } n\text{-Mahlo and } V_\theta \prec_n \mathbf{V}) : n \in \omega \}$
- $\Phi_0 := \{ \exists \theta (\theta \text{ is } n\text{-Mahlo}) : n \in \omega \}$.
- *Over ZF, Φ and Φ_0 are equivalent.*
- **Motto:** Φ allows infinite set theory to catch up with finite set theory, vis-à-vis Model Theory.

A KEY EQUIVALENCE

- **Theorem.**

1. *If $(\mathfrak{M}, \mathcal{A}) \models GBC + \text{“Ord is weakly compact”}$, then $\mathfrak{M} \models ZFC + \Phi$.*
2. *Every consistent completion of $ZFC + \Phi$ has a countable model which has an expansion to a model of $GBC + \text{“Ord is weakly compact”}$.*

[Schmerl-Shelah (1972) \rightsquigarrow Kaufmann (1983) \rightsquigarrow E(1987, 2004)]

CONCLUDING CONSIDERATIONS

- If Replacement(I) is weakened to Separation(I) in $ZFC(I)$, while retaining “ I is cofinal”, then the resulting theory is conservative over ZFC .
- We can strengthen $ZFC(I)$ to $ZFC(I^+)$ with “ C is a cub” to ensure that when the indiscernibles are stretched, a model with a *least new ordinal* is obtained.
- $ZFC(I^+)$ turns out to be a conservative extension of a $ZFC + \Psi$, where the scheme Ψ is obtained from Φ by replacing “ n -Mahlo” by “ n -subtle”, i.e., the axioms of Ψ are of the form

“ $\exists \theta (\theta \text{ is } n\text{-subtle and } V_\theta \prec_n \mathbf{V})$ ”.