

Model Theory of the Reflection Scheme

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- The reflection principle for ZF : for any set theoretical formula $\varphi(x)$ (possibly with parameters) there is a rank initial segment V_α of the universe that is φ -reflective, i.e., for any $s \in V_\alpha$, $\varphi(s)$ holds in the universe iff $\varphi(s)$ holds in V_α .
- Given a language \mathcal{L} with a distinguished symbol $<$ for a linear order, the reflection scheme over \mathcal{L} , denoted $REF(\mathcal{L})$, consists of the sentence “ $<$ is a linear order without a last element” plus the universal closure of formulas of the form

$$\exists x \forall y_1 < x \cdots \forall y_1 < x$$

$$\varphi(y_1, \cdots, y_n, v_1, \cdots, v_r) \leftrightarrow$$

$$\varphi^{<x}(y_1, \cdots, y_n, v_1, \cdots, v_r).$$

- The *regularity scheme* $REG(\mathcal{L})$ consists of the sentence “ $<$ is a linear order with no last element” plus the universal closure of axioms of the form

$$[\forall x \exists y < z \varphi(x, y, v_1, \dots, v_r)] \rightarrow$$

$$[\exists y < z \forall v \exists x > v \varphi(x, y, v_1, \dots, v_r)].$$

- Note that every model of $REF(\mathcal{L})$ is also a model of $REG(\mathcal{L})$ (but not vice versa).

• Examples

1. If κ is a regular infinite cardinal, then every expansion of a κ -like linear order satisfies the regularity scheme.
2. If κ is an *uncountable* regular cardinal and $<$ is the natural order on κ , then every expansion of $(\kappa, <)$ satisfies the reflection scheme.
3. More generally, if (X, \triangleleft) is a κ -like linear order that continuously embeds a stationary subset of κ , then any expansion of (X, \triangleleft) satisfies the reflection scheme.
4. All instances of $REG(\mathcal{L}_{PA})$ are provable in PA , where \mathcal{L}_{PA} is the language of PA . In this context $REG(\mathcal{L}_{PA})$ plus the scheme $I\Delta_0$ of bounded induction is known to be equivalent to PA .

5. ZF plus “all sets are ordinal-definable” ($V = OD$) proves all instances of the reflection scheme in the language of $\{<_{OD}, \in\}$, where $<_{OD}$ is the canonical well-ordering of the ordinal-definable sets.

6. $ZF \setminus \{\text{Power Set Axiom}\}$ plus “all sets are constructible” ($V = L$) proves all instances of the reflection scheme in the language $\mathcal{L} = \{<_L, \in\}$, where $<_L$ is the canonical well-ordering of the constructible universe.

7. The theory T of pure linear orders with no maximum element proves every instance of $REG(\{<\})$.

- **Theorem** (Keisler) *The following are equivalent for a complete first order theory T formulated in the language \mathcal{L} .*

(1) *Some model of T has an e.e.e.*

(2) *T proves $REG(\mathcal{L})$.*

(3) *Every countable model of T has an e.e.e.*

(4) *Every countable model of T has an ω_1 -like e.e.e.*

(5) *T has a κ -like model for some regular cardinal κ .*

- **Remarks**

1. In part (5) of Keisler's Theorem, κ cannot in general be chosen as ω_2 .
2. By the MacDowell-Specker Theorem, every model of PA has an e.e.e. In contrast, it is known that every completion of ZFC has an ω_1 -like model that does not have an e.e.e.
3. There is a recursive scheme Φ in the language of set theory such that: (a) every completion of $ZFC + \Phi$ has a θ -like model for any uncountable $\theta \geq \omega_1$, and (b) it is consistent (relative to $ZFC +$ "there is an ω -Mahlo cardinal") that the only completions of ZFC that have an ω_2 -like model are those that satisfy Φ .

4. Rubin refined (2) \Rightarrow (3) of Keisler's Theorem by showing that for any countable linear order \mathbb{L} , and any countable model \mathfrak{M}_0 of $REG(\mathcal{L})$ with definable Skolem functions, there is an elementary extension $\mathfrak{M}_{\mathbb{L}}$ of \mathfrak{M}_0 such that the lattice of intermediate submodels $\{\mathfrak{M} : \mathfrak{M}_0 \preceq \mathfrak{M} \preceq \mathfrak{M}_{\mathbb{L}}\}$ (ordered under \prec) is isomorphic to the Dedekind completion of \mathbb{L} .

5. Since there are continuum many nonisomorphic countable Dedekind complete linear orders, this shows that every countable complete Skolemized extension of $REG(\mathcal{L})$ has continuum many countable nonisomorphic models.

- **Theorem** *Suppose T is a consistent theory formulated in the language \mathcal{L} such that T proves $REG(\mathcal{L})$.*
1. (Chang) *If κ is a regular cardinal satisfying $\kappa^{<\kappa} = \kappa$, then T has a κ^+ -like model.*
 2. (Jensen) *If κ is a singular strong limit cardinal and \square_κ holds, then T has a κ^+ -like model.*

Remarks

1. The converse of Chang's Theorem is false.
2. Chang's Theorem has been recently revisited in the work of Villegas-Silva, who has employed the existence of a coarse $(\kappa, 1)$ -morass (instead of $\kappa^{<\kappa} = \kappa$) to establish the conclusion of Chang's Theorem for theories T formulated in languages of cardinality κ .
3. Shelah has isolated a square principle (denoted $\square_{\kappa}^{b^*}$) that is *equivalent* to the two-cardinal transfer principle $(\omega_1, \omega) \rightarrow (\kappa^+, \kappa)$.

New Results

- **Theorem** (Splitting Theorem). *Suppose $\mathfrak{M} \models REG(\mathcal{L})$ with $\mathfrak{M} \prec \mathfrak{N}$. Let \mathfrak{M}^* be the submodel of \mathfrak{M} whose universe M^* is the convex hull of M in \mathfrak{N} , i.e.,*

$$M^* := \{x \in N : \exists y \in M (x <_N y)\}.$$

Then

$$\mathfrak{M} \preceq_{cof} \mathfrak{M}^* \preceq_e \mathfrak{N}.$$

- Suppose \mathfrak{M} is a model with definable Skolem functions. \mathfrak{M} is *tall* iff for every element $c \in M$, the submodel generated by c is bounded in \mathfrak{M} .

- **Theorem** *The following three conditions are equivalent for a model \mathfrak{M} of $REG(\mathcal{L})$ with definable Skolem functions.*

(1) \mathfrak{M} is tall.

(2) \mathfrak{M} can be written as an e.e.e. chain with no last element.

(3) \mathfrak{M} has a cofinal recursively saturated elementary extension.

- **Theorem** *Every tall model of $REG(\mathcal{L})$ has a cofinal resplendent elementary extension.*

- Suppose \mathfrak{M} and \mathfrak{N} are structures with a distinguished linear order $<$, and \mathfrak{M} is a submodel of \mathfrak{N} . \mathfrak{N} is said to be a *blunt extension* of \mathfrak{M} if the supremum of M in $(N, <_N)$ exists, i.e., if $\{x \in N : \forall m \in M (m <_N x)\}$ has a first element.
- **Theorem.** *Suppose \mathfrak{M} is a resplendent model of $REG(\mathcal{L})$. Then there is some $\mathfrak{M}_0 \prec_e \mathfrak{M}$ such that $\mathfrak{M}_0 \cong \mathfrak{M}$. Moreover, if \mathfrak{M} is a model of $REF(\mathcal{L})$, then we can further require that $\mathfrak{M}_0 \prec_e^{blunt} \mathfrak{M}$.*
- **Corollary** *Every tall model of $REF(\mathcal{L})$ has a blunt elementary extension. In particular, every model of $REF(\mathcal{L})$ of uncountable cofinality has a blunt elementary extension.*

- **Theorem A.** *The following are equivalent for a complete first order theory T formulated in the language \mathcal{L} with a distinguished linear order.*

(1) *Some model of T has a blunt e.e.e.*

(2) *$T \vdash REF(\mathcal{L})$.*

(3) *Every countable recursively saturated countable model of T has a blunt recursively saturated e.e.e.*

(4) *T has an ω_1 -like e.e.e. that continuously embeds ω_1 .*

(5) *T has a κ -like model for some regular uncountable cardinal κ that continuously embeds a stationary subset of κ .*

(6) *T has a κ -like model for some regular uncountable cardinal κ that has a blunt elementary extension.*

- **Remarks**

1. In contrast with part (3) of Keisler's Theorem , it is not true in general that a countable model of the reflection scheme has a blunt e.e.e. For example, no e.e.e. of the Shepherdson-Cohen minimal model of set theory can be blunt.
2. A number of central results about stationary logic $L(aa)$ can be derived, via the 'reduction method', as corollaries of Theorem A. In particular, the countable compactness of $L(aa)$, as well as the recursive enumerability of the set of valid sentences of $L(aa)$ can be directly derived from Theorem A.

Analogue of Chang's Two-cardinal Theorem

- **Theorem B.** *Suppose T is a consistent theory containing $REF(\mathcal{L})$, and κ is a regular cardinal with $\kappa = \kappa^{<\kappa}$. Then T has a κ^+ -like model that continuously embeds the stationary subset $\{\alpha < \kappa^+ : cf(\alpha) = \kappa\}$ of κ^+ .*

Open Questions

Question 1. *In the presence of the continuum hypothesis, is it true that ω_1 can be replaced by ω_2 in part (4) of Theorem A?*

Question 2. *Let $\kappa \rightarrow_{c.u.b.} \theta$ abbreviate the transfer relation “every sentence with a κ -like model that continuously embeds a stationary subset of κ also has a θ -like model that continuously embeds a c.u.b. subset of θ ”. Is there a model of ZFC in which the only inaccessible cardinals κ such that the transfer relation $\kappa \rightarrow_{c.u.b.} \omega_2$ holds are those cardinals κ that are n -subtle for each $n \in \omega$?*

- Notice that Theorem A implies that for all regular uncountable cardinal κ , $\kappa \rightarrow_{c.u.b.} \omega_1$. To motivate this question, first let $\kappa \rightarrow \theta$ abbreviate “every sentence with a κ -like model also has a θ -like model”. The following three results suggest that Question 2 might have a positive answer:

(1) Schmerl and Shelah showed that $\kappa \rightarrow \theta$ holds for $\theta \geq \omega_1$, if κ is n -Mahlo for each $n \in \omega$;

(2) Schmerl proved that (relative to the consistency of an ω -Mahlo cardinal) there is a model of ZFC in which the only inaccessible cardinals κ such that $\kappa \rightarrow \omega_2$ holds are precisely those inaccessible cardinals κ that are n -Mahlo for each $n \in \omega$; and

(3) Schmerl established that $\kappa \rightarrow_{c.u.b.} \theta$ holds for all $\theta \geq \omega_1$ if κ is n -subtle for each $n \in \omega$.

Question 3. *Can Theorem B be strengthened by (1) weakening the hypothesis $\kappa = \kappa^{<\kappa}$ to Shelah's square principle $\square_{\kappa}^{b^*}$ (mentioned in Remark 1.6.1), or (2) by using coarse $(\kappa, 1)$ morasses so as to allow T to have cardinality κ ?*

Question 4. *Let $<$ be the natural order on ω^ω and suppose (A, \triangleleft) and $(\omega^\omega, <)$ are elementarily equivalent. Does (A, \triangleleft) have a blunt e.e.e.?*

- By a classical theorem of Ehrenfeucht,

$$(\omega^\omega, <) \prec (\text{Ord}, <).$$

The answer to Question 4 is unknown even when A is countable.